On the distance between separatrices for the discretized logistic differential equation

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abstract

In this paper, we consider the discretization

$$y(t + \varepsilon) = y(t - \varepsilon) + 2\varepsilon\big(1 - y(t)^2\big),$$

$\varepsilon > 0$ a small parameter, of the logistic differential equation $y' = 1 - y^2$, which can also be seen as a discretization of the system

$$\begin{cases} y' = 2\big(1 - v^2\big), \\ v' = 2\big(1 - y^2\big). \end{cases}$$

This system has two saddle points at $A = (1, 1)$, $B = (-1, -1)$ and there exist stable and unstable manifolds. We will show that the stable manifold W_s^+ of the point $A = (1, 1)$ and the unstable manifold W_i^- of the point $B = (-1, -1)$ for the discretization do not coincide. The vertical distance between these two manifolds is exponentially small but not zero, in particular we give an asymptotic estimate of this distance. For this purpose we will use a method adapted from the paper of Schäfke-Volkmer [5] using formal series and accurate estimates of the coefficients.

Keywords : Difference equation ; Manifolds ; Linear operator ; Formal solution ; Gevrey asymptotic ; Quasi-solution

1 Introduction

We consider the logistic equation

$$y' = 1 - y(t)^2, \tag{1.1}$$

whose solutions are $y(t) = \tanh(t + c)$. The discretization of this equation by Nyströrm's method, which consists in replacing the derivative by the symmetrical difference, gives the recurrence :

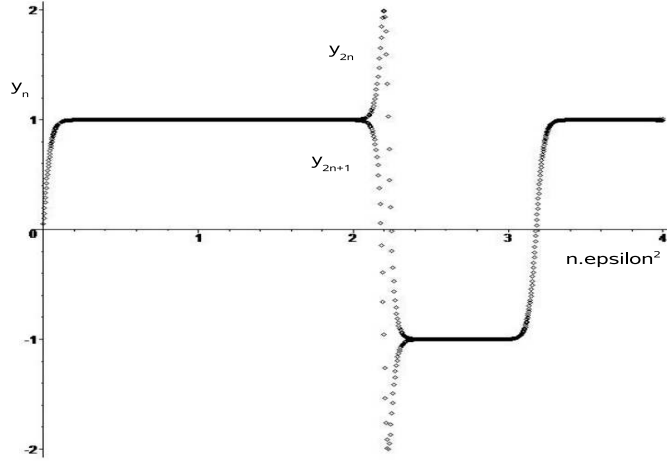$$y_{n+1} = y_{n-1} + 2\varepsilon\big(1 - y_n^2\big). \tag{1.2}$$

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FIG. 1 – This figure represents y_n as a function of $n\varepsilon^2$.

With the initial conditions $y_0 = 0$, $y_1 = \varepsilon$, we calculate the discrete solution y_n with 1600 iterates and $\varepsilon = \frac{1}{20}$ (see FIG.1). We observe that the discrete solution rejoins quickly the level $y = 1$, it stays close to this point for a relatively long time before it leaves its neighborhood, then for the little while the even numbered points follow one curve and the odd numbered follow another curve, the two curves meet at the level $y = -1$, the discrete solution stays close to this point for a relatively long time, then it remakes the same cycle.

Now letting $u_n = y_{2n}$, $v_n = y_{2n+1}$, we obtain a recurrence of first order in the plane

$$(u_{n+1}, v_{n+1}) = \Phi(u_n, v_n) \tag{1.3}$$

where the diffeomorphism $\Phi : \mathrm{IR}^2 \mapsto \mathrm{IR}^2$ is defined by

$$
\begin{aligned}
u_1 &= u + 2\varepsilon\big(1 - v^2\big), \\
v_1 &= v + 2\varepsilon\big(1 - u_1^2\big).
\end{aligned}
$$

This is a discretization of the following system of differential equations

$$
\begin{cases}
u' = 2\big(1 - v^2\big), \\
v' = 2\big(1 - u^2\big).
\end{cases} \tag{1.4}
$$

We notice easily that the set $\mathfrak{E} = \big\{(u, v) \backslash (u - v)(u^2 + uv + v^2 - 3) = 0\big\}$ is an invariant set for this system. The system (1.4) has two saddle points in $A = (1, 1)$

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and $B = (-1, -1)$ and corresponding stable and unstable manifolds lie on its invariant set $\mathfrak{E}$.

The stable manifold at A and unstable manifolds at B are part of set $\{(u, v)\backslash u = v\}$, where as the unstable manifold at A and stable manifolds at B are part of set $\{(u, v)\backslash u^2 + uv + v^2 = 3\}$. The stable manifold of A coincides with the unstable manifold of B (See FIG.2).

For the discretized equation (1.3), these manifolds still exist [2], let W_s^+, W_i^- denote the stable manifold at A and the unstable manifold at B respectively, W_i^+ and W_s^- unstable manifold at A and stable manifold at B respectively, W_s^+ and W_i^- do not coincide an more (See FIG.3) as we want to show.

In the paper [2], after introducing the notion of length of the first level $l_1(\varepsilon) = 2\varepsilon n_1(\varepsilon)$, where $n_1(\varepsilon) = \inf\{n \in \text{IN}\backslash y_{2n}(\varepsilon) + y_{2n+1}(\varepsilon) < 0\}$, Fruchard-Schäfke had shown that there exist a postive constant K such that

$$\Delta(\varepsilon) \leq \exp\left(-\frac{\pi^2 + o(1)}{2\varepsilon}\right), \quad \text{as } \varepsilon \searrow 0,$$

$$l_1(\varepsilon) \geq \frac{\pi^2}{4\varepsilon} + o(1),$$

where $\Delta(\varepsilon)$ denote the distance between the sets $W_s^+ \cap S$ and $W_i^- \cap S$ in the sense of Hausdorff and $S = \{(u, v)\backslash -1 \leq u + v \leq 1\}$.

With the initial condition $y_0 = 0$, $y_1 = \varepsilon$, the length of the first level satisfies $l_1(\varepsilon) = -\frac{1}{2}(1 + o(1)) \log(\Delta(\varepsilon))$ as $\varepsilon \searrow 0$[2]. They also showed that there are two families of entire functions $y_\varepsilon^\pm : \mathbb{C} \longmapsto \mathbb{C}$, solutions of the difference equation

$$y(t + \varepsilon) = y(t - \varepsilon) + 2\varepsilon(1 - y(t)^2), \tag{1.5}$$

and the functions $t \mapsto (y_\varepsilon^+(t), y_\varepsilon^+(t + \varepsilon))$, $t \mapsto (y_\varepsilon^-(t), y_\varepsilon^-(t + \varepsilon))$ provide parametrization $w_s^+(t)$ of W_s^+ for $t \in [-1, \infty[$ respectively $w_i^-(t)$ of W_i^- for $t \in]-\infty, 1]$.

In this work we will prove

Theorem 1.1. *There exist a constant α with $1.2641497 \leq \alpha \leq 1.2641509$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

$$Dist_\varepsilon\left(w_s^+(t), W_i^-\right) = \frac{4\pi\alpha \cos(\frac{\pi}{\varepsilon}t + \pi)}{\varepsilon^3(1 - \tanh(t)^2)}e^{-\frac{\pi^2}{2\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon^2}e^{-\frac{\pi^2}{2\varepsilon}}\right), \quad as \ \varepsilon \searrow 0,$$

where $Dist_\varepsilon$ is defined as the vertical distance between the stable and unstable manifolds.

In order to show the theorem 1.1, we start with the construction of a formal power series solution in d whose coefficients are polynomials in $u = \tanh(dt/\varepsilon)$, afterwards we will give asymptotic approximations of these coefficients using appropriate norms on the spaces of polynomials. The next step is to construct a quasi-solution i.e. a function that satisfies the equation (1.5) except for an exponentially small error, then we show that this quasi-solution and the exact solution of equation (1.5) are exponentially close.
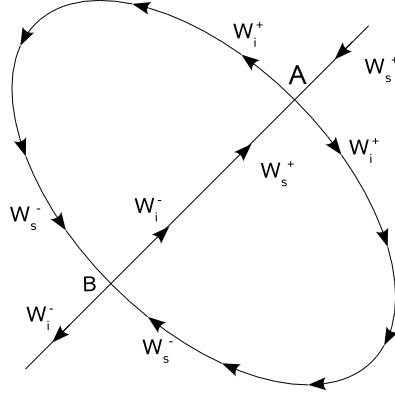
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F_{IG}. 2 – The stable and unstable manifolds for the logistic differential equation.

Finally we give an asymptotic estimate of the distance between the stable manifold of A and the unstable manifold of B and we show that this distance is exponentially small but not zero, thus completing the proof of the theorem 1.1.
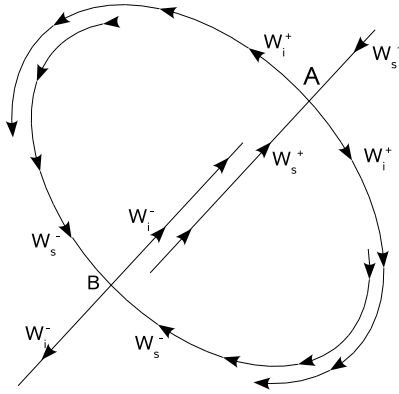

F_{IG}. 3 – The stable and unstable manifolds for the difference equation.

2 Formal solutions

We consider the difference equation

$$y(t + \varepsilon) = y(t - \varepsilon) + 2\varepsilon\big(1 - y(t)^2\big), \qquad (2.1)$$

where $\varepsilon > 0$ is the discretization step and $y(t) \to 1$ as $t \to +\infty$. Our first aim is to transform (2.1) in such a way that the new equation admits a formal solutions whose coefficients are polynomials. We define u and $A(d, u)$ such that

$$u := \tanh(\frac{d}{\varepsilon}t), \qquad (2.2)$$

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$$y_\varepsilon(t) = A(d, u), \tag{2.3}$$

where $d := \sum_{n=1}^{+\infty} d_n \varepsilon^n$ is a series to be determined. When these variables are substituted in (2.1), the following equation satisfied by $A(d, u)$ is obtained :

$$A\big(d, T^+(d, u)\big) - A\big(d, T^-(d, u)\big) = f\big(\varepsilon, A(d, u)\big), \tag{2.4}$$

where

$$f(\varepsilon, x) := 2\varepsilon(1 - x^2), \tag{2.5}$$

$$T^+ := T^+(d, u) = \frac{u + \tanh(d)}{1 + u \tanh(d)}, \tag{2.6}$$

$$T^- := T^-(d, u) = \frac{u - \tanh(d)}{1 - u \tanh(d)}. \tag{2.7}$$

For small ε one can construct a formal expansion in powers of ε^2 of the form $\sum_{n=0}^{\infty} A_{2n+1}(u) d^{2n}$, where A_{2n+1} are polynomials which satisfy $A_1(u) = u$ and $A_{2n+1}(\pm 1) = 0$, for $n \geq 1$.

The existence of a such formal solution is only possible if d and ε are coupled in a very special way. Indeed, suppose that there exists a such formal power series solution. We differentiate (2.4) with respect to u and obtain at $u = 1$

$$\frac{\partial A}{\partial u}(d, 1) \cdot \lim_{u \to 1} \frac{T^+(d, u) - 1}{u - 1} - \frac{\partial A}{\partial u}(d, 1) \cdot \lim_{u \to 1} \frac{T^-(d, u) - 1}{u - 1} = \frac{\partial f}{\partial x}(\varepsilon, 1) \cdot \frac{\partial A}{\partial u}(d, 1).$$

Because $A_{2n+1}(1) = 0$ for $n \geq 1$ implies $A(d, 1) = 1$. Thus with $\frac{\partial f}{\partial x}(\varepsilon, 1) = -4\varepsilon$ and $\frac{\partial A}{\partial u}(d, 1) = 1 + O(d) \neq 0$, the above equation implies

$$\varepsilon = \frac{\tanh(d)}{1 - \tanh(d)^2} = \frac{1}{2}\sinh(2d). \tag{2.8}$$

Theorem 2.1. *Let ε and d be coupled by $\varepsilon = \frac{1}{2}\sinh(2d)$. Then (2.4) has a formal power series solution that can be written in the form*

$$A(d, u) = u + \sum_{n=1}^{\infty} A_{2n+1}(u) d^{2n}, \tag{2.9}$$

where $A_{2n+1}(u)$ are odd polynomial and $A_{2n+1}(1) = A_{2n+1}(-1) = 0$ for all n.

Proof. We will use the Induction Principle for even n to show that there exist unique odd polynomials $A_1, A_3, A_5...A_{n+1}$ such that

$$Z(d, u) = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} A_{k+1} d^k, \tag{2.10}$$

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satisfy

$$Z\big(d, T^+(d, u)\big) - Z\big(d, T^-(d, u)\big) = f\big(\varepsilon, Z(d, u)\big) \ mod \ d^{n+2}. \tag{2.11}$$

For $n = 0$, we put $A_1(u) = u$ and $Z(d, u) = u$ and obtain

$$Z\big(d, T^+(d, u)\big) - Z\big(d, T^-(d, u)\big) = T^+(d, u) - T^-(d, u) = (2 - 2u^2)d + \mathcal{O}(d^3).$$

and

$$f\big(\varepsilon, Z(d, u)\big) = 2\varepsilon(1 - u^2) = (2 - 2u^2)d + \mathcal{O}(d^3).$$

This gives

$$Z\big(d, T^+(d, u)\big) - Z\big(d, T^-(d, u)\big) = f\big(\varepsilon, Z(d, u)\big) \ mod \ d^2.$$

Now suppose that for some even n already $A_1, A_3, A_5...A_{n+1}$ have been found with the above properties. We have to construct A_{n+3}. First, we show that $Z(d, u)$ satisfies (2.11) even modulo d^{n+3}. To this purpose, let

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = f\big(\varepsilon, Z(d, u)\big) + R_{n+2}(u)d^{n+2} + \mathcal{O}(d^{n+3}) \tag{2.12}$$

we replace d by $-d$. Using that Z is even and ε is odd in d, we obtain

$$Z\big(d, T^-\big) - Z\big(d, T^+\big) = -2\varepsilon\big(1 - Z(d, u)^2\big) + R_{n+2}(u)d^{n+2} + \mathcal{O}(d^{n+3}),$$

this gives

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = 2\varepsilon(d)\big(1 - Z(d, u)^2\big) - R_{n+2}(u)(d)^{n+2} + \mathcal{O}(d^{n+3}).$$

With (2.12) this implies $R_{n+2}(u) = 0$ and consequently

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = f\big(\varepsilon, Z(d, u)\big) + R_{n+3}(u)d^{n+3} + \mathcal{O}(d^{n+4}). \tag{2.13}$$

We want to construct $A_{n+3}(u)$ such that

$$\widetilde{Z}\big(d, T^+\big) - \widetilde{Z}\big(d, T^-\big) = f\big(\varepsilon, \widetilde{Z}(d, u)\big) + \mathcal{O}(d^{n+4}), \tag{2.14}$$

if we put

$$\widetilde{Z}(d, u) = Z(d, u) + A_{n+3}(u)d^{n+2}. \tag{2.15}$$

To this purpose we use again Taylor expansion

$$
\begin{aligned}
f\big(\varepsilon, \widetilde{Z}(d, u)\big) &= f\big(\varepsilon, Z(d, u)\big) - 4u\, A_{n+3}(u)d^{n+3} + &&\mathcal{O}(d^{n+4}), \\
\widetilde{Z}\big(d, T^+(d, u)\big) &= Z(d, u) + A_{n+3}(u)d^{n+2} + (1 - u^2)\frac{\partial A_{n+3}}{\partial u}(u)d^{n+3} + \mathcal{O}(d^{n+4}), \\
\widetilde{Z}\big(d, T^-(d, u)\big) &= Z(d, u) + A_{n+3}(u)d^{n+2} - (1 - u^2)\frac{\partial A_{n+3}}{\partial u}(u)d^{n+3} + \mathcal{O}(d^{n+4}).
\end{aligned}
$$

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With (2.13), this gives

$$\widetilde{Z}(d,T^+) - \widetilde{Z}(d,T^-) = f\big(\varepsilon,\widetilde{Z}(d,u)\big) + \widetilde{R}_{n+3}(u)d^{n+3} + \mathcal{O}(d^{n+4}), \qquad (2.16)$$

where

$$\widetilde{R}_{n+3}(u) = 2(1-u^2)\frac{\partial A_{n+3}}{\partial u}(u) + 4u\, A_{n+3}(u) + R_{n+3}(u).$$

We see that (2.14) is satisfied if only if

$$2(1-u^2)\frac{\partial A_{n+3}}{\partial u}(u) + 4u\, A_{n+3}(u) + R_{n+3}(u) = 0. \qquad (2.17)$$

This equation has a unique odd solution which is given by

$$A_{n+3}(u) = -(1-u^2)\int_0^u \frac{R_{n+3}(t)}{2(1-t^2)^2}dt.$$

We can prove that this solution is polynomial. For that, it is necessary that $R_{n+3}(u)$ and $R'_{n+3}(u)$ vanish into 1 and -1. Indeed, if we take (2.13) with $u = 1$, we obtain

$$R_{n+3}(1)d^{n+3} = Z\big(d,T^+(d,1)\big) - Z\big(d,T^-(d,1)\big) - f\big(\varepsilon, Z(d,1)\big) + \mathcal{O}(d^{n+4}). \quad (2.18)$$

Since, $T^+(d,1) = T^-(d,1) = Z(d,1) = 1$ and $f\big(\varepsilon,1\big) = 0$, we obtain

$$R_{n+3}(1)d^{n+3} = \mathcal{O}(d^{n+4}), \qquad (2.19)$$

therefore $R_{n+3}(1) = 0$. In order to show that $R'_{n+3}(1) = 0$, we derive formally (2.13) and replace u by 1. Using $T^+(d,1) = T^-(d,1) = Z(d,1) = 1$ and $\frac{\partial f}{\partial u}\big(\varepsilon,1\big) = -4\varepsilon$, we obtain

$$R'_{n+3}(1)d^{n+3} = \frac{\partial Z}{\partial u}(d,1)\left(\frac{\partial T^+}{\partial u}(d,1) - \frac{\partial T^-}{\partial u}(d,1) + 4\varepsilon\right) + \mathcal{O}(d^{n+4}). \qquad (2.20)$$

By choice of d, cf (2.8). Thus we have $\frac{\partial T^+}{\partial u}(d,1) - \frac{\partial T^-}{\partial u}(d,1) = -2\sinh(2d)$. with (2,8) we obtain

$$R'_{n+3}(1)d^{n+3} = \mathcal{O}(d^{n+4}), \qquad (2.21)$$

therefore $R'_{n+3}(1) = 0$. As formula (2.13) shows that $R_{n+3}(u)$ is odd, then $R_{n+3}(-1) = R'_{n+3}(-1) = 0$.

The first polynomials $A_{n+1}(u)$ with even n are given by

n	0	2	4	6
$A_{n+1}(u)$	u	$u - u^3$	$(1-u^2)\left(\frac{4}{3}u - \frac{10}{3}u^3\right)$	$(1-u^2)\left(\frac{62}{3}u^5 - \frac{190}{9}u^3 + \frac{182}{45}u\right)$

We introduce the operators $\mathcal{C}_2, \mathcal{C}, \mathcal{S}_2, \mathcal{S}$ defined by

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$$\mathcal{C}(Z)(d,u) = \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) + Z(d,T^{-\frac{1}{2}})\big),$$

$$\mathcal{S}(Z)(d,u) = \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) - Z(d,T^{-\frac{1}{2}})\big),$$

$$\mathcal{C}_2(Z)(d,u) = \tfrac{1}{2}\big(Z(d,T^{+}) + Z(d,T^{-})\big),$$

$$\mathcal{S}_2(Z)(d,u) = \tfrac{1}{2}\big(Z(d,T^{+}) - Z(d,T^{-})\big).$$

(2.22)

where $T^{+\frac{1}{2}} = T^{+}(\frac{d}{2},u)$, $T^{-\frac{1}{2}} = T^{-}(\frac{d}{2},u)$ and $Z(d,u)$ is a formal power series of d whose coefficients are polynomials. We rewrite equation (2.4) as

$$\mathcal{S}_2(A)(d,u) = \varepsilon\big(1 - A(d,u)^2\big). \qquad (2.23)$$

3   Norms for polynomials and basis

In the sequel we denote :
- $\mathcal{P}$ the set of all polynomial whose coefficents are complex,
- $\mathcal{P}_n$ the spaces of all polynomials of degree less than or equal to n,
- $\mathcal{Q} := \{Q(d,u) = \sum_{n=0}^{\infty} Q_n(u)d^n,\ \text{where } Q_n(u) \in \mathcal{P}_n,\ \text{for all } n \in \mathbb{N}\}$.

Proposition 3.1. *[5] If we define the sequence of the polynomial functions $\tau_n(u)$ by $\tau_0(u) = 1, \tau_1(u) = u, \tau_{n+1}(u) = \frac{1}{n}D\tau_n(u)$, where the operator D is defined by $D := (1 - u^2)\frac{\partial}{\partial u}$, we have*

1. *$T^{+}(d,u) = \sum_{n=0}^{\infty} \tau_{n+1}(u)d^n$,*

2. *$\tau_n(u)$ has exactly degree n,*

3. *$\tau_n(\tanh(z)) = \frac{1}{(n-1)!}\big(\frac{d}{dz}\big)^{n-1}\big(\tanh(z)\big)$.*

Definition 3.2. *Let $p \in \mathcal{P}_n$. As $\tau_0(u), \tau_1(u), ..., \tau_n(u)$ form a basis of $\mathcal{P}_n$, we can write p as*

$$p = \sum_{k=0}^{n} a_k \tau_k(u).$$

Then we define the norm

$$\|p\|_n = \sum_{i=0}^{n} a_i \left(\frac{\pi}{2}\right)^{n-i}. \qquad (3.1)$$

Theorem 3.3. *[5] Let n,m be positive integers and $p \in \mathcal{P}_n$, $q \in \mathcal{P}_m$. The norms $\|\ \|_n$ of the above definition have the following property :*

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1. $\|Dp\|_{n+1} \leq n\|p\|_n$.

2. If p odd we have $\|p\|_n \leq \|Dp\|_{n+1}$.

3. There exists a constant M_1 such that $\|pq\|_{n+m} \leq M_1\|p\|_n|q\|_m$.

4. There is a constant M_2 such that $|p(u)| \leq M_2\left(\frac{2}{\pi}\right)^n\|p\|_n$ $(-1 \leq u \leq 1)$.

5. There is a constant M_3 such that for all $n > 1$ with $p(-1) = p(-1) = 0$,

$$\left\|\frac{p}{\tau_2}\right\|_{n-2} \leq M_3\|p\|_n.$$

4   Operators

In this section we will use definitions and results adapted from [5] by replacing $\frac{\pi}{2}$ by π.

Definition 4.1. *Let f be formal power series of z whose coefficients are complex. We define a linear operator $f(dD)$ on $\mathcal{Q}$ by*

$$f(dD)Q = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} f_i D^i Q_{n-i}(u)\right)d^n, \tag{4.1}$$

Where $f(z) = \sum_{i=0}^{\infty} f_i z^i$ and $Q = \sum_{n=0}^{\infty} Q_n(u)d^n \in \mathcal{Q}$.

By the above definition and (1) of the proposition 3.1, we can show that

$$Q(d, T^+(\theta d, u)) = \big(\exp(\theta dD)Q\big)(d, u), \quad \text{for} \;\; Q \in \mathcal{Q} \text{ and all } \theta \in \mathbb{C}. \tag{4.2}$$

With (2.22) this implies

$$\begin{aligned}
\mathcal{C}(Q)(d, u) &= \cosh(\tfrac{d}{2}D)Q, \;\; \mathcal{S}(Q)(d, u) &= \sinh(\tfrac{d}{2}D)Q, \\
\mathcal{C}_2(Q)(d, u) &= \cosh(dD)Q, \;\; \mathcal{S}_2(Q)(d, u) &= \sinh(dD)Q,
\end{aligned}$$

for polynomial series Q in $\mathcal{Q}$. We denote $\|Q\|_n = \|Q_n\|_n$.

Theorem 4.2. *. Let $f(z)$ be formal power series having a radius of convergence greater than π and let k be a positive integer. There is a constant K such that : If Q is a polynomial series having the following property*

$$\|Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ M(n-k)!\pi^{-n} & \text{for } n \geq k \end{cases}$$

where M independent of n and $Q \in \mathcal{Q}$ then the polynomial series $f(dD)Q$ satisfies

$$\|f(dD)Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ MK(n-k)!\pi^{-n} & \text{for } n \geq k \end{cases}$$

We define the operator $\mathcal{C}^{-1}$ by $\mathcal{C}^{-1} = g(dD)$, where $g(z) = \dfrac{1}{\cosh(z/2)}$

Theorem 4.3. *There exists a positive constant K such that, if Q is a polynomial series such that Q_n odd, $\|Q\|_n = 0$ for $n < k$ for some positive integer k and*

$$\|dDQ\|_n \leq M(n-k)\,!\,\pi^{-n} \quad \text{for } n \geq k$$

where M independent of n and $Q \in \mathcal{Q}$, the polynomial series $\mathcal{C}^{-1}(Q)$ satisfies

$$\left\|\mathcal{C}^{-1}(Q)\right\|_n \leq MK\pi^{-n} \begin{cases} n\,! & \text{for } k = 1 \\ (n-1)\,!\,\log(n) & \text{for } k = 2 \\ (n-1)\,! & \text{for } k \geq 3 \end{cases}$$

Theorem 4.4. *We consider a polynomial series*

$$Q_\alpha(d, u) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \alpha_n (n-1)\,!\left(\frac{i}{\pi}\right)^{n-1} \tau_n(u) d^n$$

where $\alpha_n = \mathcal{O}(n^{-k})$ as $n \to \infty$ with some integer $k \geq 2$. Let $\alpha := \frac{1}{\pi}\sum_{n=1}^{\infty} \alpha_n$. then the coefficients $\left\{\mathcal{C}^{-1}(Q_\alpha)\right\}_n$ of $\mathcal{C}^{-1}(Q_\alpha)$ satisfy

$$\left\| \left\{\mathcal{C}^{-1}(Q_\alpha)\right\}_n - \alpha(n-1)\,!\left(\frac{i}{\pi}\right)^{n-1}\tau_n \right\|_n = \mathcal{O}\!\left((n-k)\,!\,\pi^{-n}\right),$$

as $n \to \infty$ for odd n.

Theorem 4.5. *Let k, l, p, q be positive integer with $p \geq k$ and $q \geq l$. Define m as the minimum of $k + q$ and $l + p$. Then there is a constant K with the following property :*

If P and Q are polynomial series such that $\|P\|_n = 0$ for $n < p$, $\|Q\|_n = 0$ for $n < q$ and

$$\begin{aligned}\|P\|_n &\leq M_1(n-k)\,!\,\pi^{-n}, \quad \text{for } n \geq k, \\ \|Q\|_n &\leq M_2(n-l)\,!\,\pi^{-n}, \quad \text{for } n \geq l.\end{aligned}$$

Then

$$\|PQ\|_n \leq KM_1M_2(n-m)\,!\,\pi^{-n}, \quad \text{for } n \geq p + q.$$

Theorem 4.6. *Let $Q_1(d, u)$ be a convergent polynomial series which is even with respect to both variables and has constant term 1.*
Let $Q_2(d, u) = d^2(1 - u^2)Q_1(d, u)$ and $P(d, u) = \mathcal{S}(Q_2)/\mathcal{C}(Q_2)$. Consider the linear operator defined by

$$\mathcal{L}(Q) = \mathcal{S}(Q) - P(d, u) \cdot \mathcal{C}(Q), \quad Q \in \mathcal{Q}. \tag{4.3}$$

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Then, there exist a constant K with the following property. If Q is an odd polynomial series with odd coefficients $Q_n(u)$ satisfying $Q_n(1) = 0$ for all n, $\|\mathcal{L}(Q)\|_n = 0$ for $n < 6$ and

$$\|\mathcal{L}(Q)\|_n \leq M(n-6)\,!\,\pi^{-n} \text{ for } n \geq 6,$$

then also

$$\|dDQ\|_n \leq KM(n-6)\,!\,\pi^{-n} \text{ for } n \geq 6.$$

5   Asymptotic approximation of the coefficients of the formal solution

The objective in this section is to construct an asymptotic aproximation of the coefficents of the formal solution (2.9). It will turn out to be convenient to consider the new series $B(d,u) = \varepsilon A(d,u)$,

$$B(d,u) = \varepsilon A(d,u) = \sum_{\substack{n=2 \\ n \ even}}^{\infty} A_{n+1}(u)\varepsilon d^n + \varepsilon u,$$

this gives

$$B(d,u) = \sum_{\substack{n=3 \\ n \ odd}}^{\infty} B_n(u)d^n + \varepsilon u,$$

where $B_n(u)$ are odd polynomials. Furthermore we have $B_n(u) \in \mathcal{P}_n$ and $B_n(1) = B_n(-1) = 0$ for all n. The new equation for $B(d,u)$ is

$$\mathcal{S}_2(B)(d,u) = \varepsilon^2 - B(d,u)^2. \tag{5.1}$$

We saw in section 2 that the series A was a formal solution of the starting equation. For the moment, nothing is known about the norms of its coefficients, but we will show that this series is Gevrey-1, more precisely $\|A\|_n = O\big(n\,!\,\pi^{-n}\big)$. This enables us thereafter to construct a quasi-solution. To this purpose, we will prepare the equation (5.1) so that we can construct a recurrence.

We start with the decompostion of series $B(d,u)$ in the form :

$$B = U + F, \text{ where } U = \varepsilon u + (u - u^3)d^3 + \Big(\frac{10}{3}u^5 - \frac{16}{3}u^3 + 2u\Big)d^5. \tag{5.2}$$

Next, we define

$$\begin{aligned} G : \ &= \ QF, \\ J : \ &= \ \frac{\mathcal{C}(G)}{Q_1}. \end{aligned} \tag{5.3}$$

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where

$$Q(d, u) := (1 - u^2)d^2 + (u^4 - u^2)d^4 - \left(\frac{13}{6}u^6 - \frac{7}{2}u^4 + \frac{4}{3}u^2\right)d^6$$
$$+ \left(\frac{47}{6}u^8 - \frac{31}{6}u^6 + \frac{58}{15}u^4 - \frac{104}{45}u^2 + 1\right)d^8,$$
$$Q_1(d, u) = (1 - u^2)d^2 + \frac{3}{2}(u^2 - u^4)d^4. \tag{5.4}$$

The choice of Q and Q_1 and by using the properties of the operators $\mathcal{S}, \mathcal{C}$, we will be able to rewrite the equation (5.1) in the form :

$$e_0(d, u)\mathcal{L}(J) = e_1(d, u)\mathcal{C}(J) + e_2(d, u)F + e_3(d, u)F^2$$
$$+ e_4(d, u)\mathcal{C}_2(F) + e_5(d, u)$$

where $\mathcal{L}$ is the operator defined in (4.3) and $e_i(d, u), i = 0, ..., 5$ are convenient known convergent series in d and u which will be thereafter given.

The left hand side of this equation is series with leading term 1 multiplied by the invertible operator $\mathcal{L}$ applied to the series J. The right hand side is an expression of F and J multiplied by known convergent series $e_i(d, u), i = 1, ..., 5$.

U, Q, Q_1 were chosen so that the series $e_i(d, u), i = 0..5$ are of a rather large order in d, this makes the second term smaller than the right hand side. This property will be useful to construct a recurrence on n and to reverse then the operators $\mathcal{L}, \mathcal{C}$, which makes possible to estimate the coefficients of series $\mathcal{C}(G)$ and finally the coefficients of the formal solution of the equation (5.1).

More precisely, we insert (5.2) into the equation (5.1) and find

$$\mathcal{S}_2(F) = -F^2 - 2U F + \varepsilon^2 - U^2 - \mathcal{S}_2(U). \tag{5.5}$$

We define $\mathcal{X}$ by

$$\mathcal{X} := V(d, u) \cdot \mathcal{SC}(G) + W(d, u) \cdot \mathcal{C}^2(G) \tag{5.6}$$

where

$$V(d, u) := 1 + 2u^2 d^2 - \left(\frac{7}{3}u^4 - 3u^2\right)d^4 - \left(\frac{31}{3}u^6 - \frac{221}{9}u^4 + \frac{253}{15}u^2 - \frac{41}{15}\right)d^6,$$
$$W(d, u) := 2\varepsilon u.$$

Then, because of $G = Q F$, we have $\mathcal{X}(G) = V \cdot \mathcal{SC}(F Q) + W \cdot \mathcal{C}^2(F Q)$. Using the formulas

$$\mathcal{C}_2 = 2\mathcal{C}^2 - Id,$$

$$\mathcal{S}_2 = 2\mathcal{SC},$$

$$\mathcal{S}_2(F Q) = \mathcal{S}_2(F)\mathcal{C}_2(Q) + \mathcal{S}_2(Q)\mathcal{C}_2(F), \tag{5.7}$$

$$\mathcal{C}_2(F Q) = \mathcal{C}_2(F)\mathcal{C}_2(Q) + \mathcal{S}_2(Q)\mathcal{S}_2(F).$$

We obtain

$$
\begin{aligned}
\mathcal{X} &= \frac{1}{2} V \cdot \mathcal{S}_2(F\,Q) + \frac{1}{2} W \cdot \Big(\mathcal{C}_2(F\,Q) + Q\,F \Big) \\
&= \frac{1}{2} \Big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q) \Big) \mathcal{S}_2(F) + \frac{1}{2} \Big(V\,\mathcal{S}_2(Q) + W\,\mathcal{C}_2(Q) \Big) \mathcal{C}_2(F) \\
&\quad + \frac{1}{2} W\,Q \cdot F.
\end{aligned}
$$

With (5.5) this implies

$$
\mathcal{X} = \frac{1}{2} W_1\,F + \frac{1}{2} W_2\,\mathcal{C}_2(F) - \frac{1}{2} (V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)) F^2 + \mu(\varepsilon, u), \qquad (5.8)
$$

where

$$
W_1 = -2U \big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q) \big) + W\,Q,
$$

$$
W_2 = V\,\mathcal{S}_2(Q) + W\,\mathcal{C}_2(Q),
$$

$$
\mu(\varepsilon, u) = \tfrac{1}{2} \big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q) \big) \big(\varepsilon^2 - U^2 - \mathcal{S}_2(U) \big).
$$

The lowest power of d in W_1 and W_2 is the eleventh $W_1 = \mathcal{O}(d^{11}) = W_2$, $\mu(\varepsilon, u)$ is analytic. On the other hand $\mathcal{C}(G) = Q_1\,J$, this implies

$$
\mathcal{X}(G) = V \cdot \mathcal{S}(Q_1\,J) + W \cdot \mathcal{C}(Q_1\,J). \qquad (5.9)
$$

If we use the product formulas

$$
\begin{aligned}
\mathcal{S}(Q_1\,J) &= \mathcal{S}(J)\mathcal{C}(Q_1) + \mathcal{S}(Q_1)\mathcal{C}(J), \\
\mathcal{C}(Q_1\,J) &= \mathcal{C}(J)\mathcal{C}(Q_1) + \mathcal{S}(Q_1)\mathcal{S}(J).
\end{aligned}
\qquad (5.10)
$$

We have

$$
\mathcal{X}(G) = \Big(V\mathcal{C}(Q_1) + W\mathcal{S}(Q_1) \Big) \mathcal{S}(J) + \Big(V\mathcal{S}(Q_1) + W\mathcal{C}(Q_1) \Big) \mathcal{C}(J). \qquad (5.11)
$$

With (5.8) and (4.3), this implies

$$
\begin{aligned}
V_1 \mathcal{L}(J) &= -(W_3 + PV_1)\mathcal{C}(J) + \frac{1}{2} W_1 F - \frac{1}{2} (V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)) F^2 \\
&\quad + \frac{1}{2} W_2\,\mathcal{C}_2(F) + \mu(\varepsilon, u),
\end{aligned}
\qquad (5.12)
$$

$$
\begin{aligned}
V_1 &= V\mathcal{C}(Q_1) + W\mathcal{S}(Q_1), \\
W_3 &= V\mathcal{S}(Q_1) + W\mathcal{C}(Q_1), \\
P &= \mathcal{S}\big((1 - u^2)d^2 \big) / \mathcal{C}\big((1 - u^2)d^2 \big) = -ud + \mathcal{O}(d^2), \\
\mathcal{L}(J) &= \mathcal{S}(J) - P \cdot \mathcal{C}(J).
\end{aligned}
\qquad (5.13)
$$

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We divide (5.12) by $(1-u^2)d^2$, this implies

$$\frac{V_1}{(1-u^2)d^2}\mathcal{L}(J) = -\frac{W_3+PV_1}{(1-u^2)d^2}\mathcal{C}(J) + \frac{W_1}{2(1-u^2)d^2}F - \frac{V\,\mathcal{C}_2(Q)+W\,\mathcal{S}_2(Q)}{2(1-u^2)d^2}F^2$$
$$+ \frac{W_2}{2(1-u^2)d^2}\mathcal{C}_2(F) + \frac{\mu(\varepsilon,u)}{(1-u^2)d^2}, \tag{5.14}$$

where F and J are coupled by the equation $Q_1 J = \mathcal{C}(Q\,F) = \mathcal{C}(G)$.

Theorem 5.1. *With the above notations, we have the following estimation*

$$\big\| dD\mathcal{C}(G) \big\|_{n+1} = \mathcal{O}\Big((n-7)\,!\,\pi^{-n}\Big) \quad as\ n \to \infty.$$

Proof. We set
$$e_n := \frac{\big\| dD\mathcal{C}(G) \big\|_{n+1}}{(n-6)\,!\,\pi^{-n}} \quad \text{for}\ \ n \geq 7. \tag{5.15}$$

We must show that $e_n = \mathcal{O}(n^{-1})$. In the sequel we will use the convention : If $a_n, n = 0, 1, ...$ is any sequence of positive real numbers, then

$$a_n^+ := \max(a_0, a_1, ...a_n), \ \text{for all}\ n \geq 0. \tag{5.16}$$

In this proof $K_1, K_2, ...$ will always denote constants independent of n. Theorem 4.3 gives

$$\big\| G \big\|_n \leq K_1 e_n^+ (n-1)\,!\,\pi^{-n}, \tag{5.17}$$

$$\big\| G^2 \big\|_n \leq K_2 f_n (n-10)\,!\,\pi^{-n}, \quad \text{for}\ n \geq 18, \tag{5.18}$$

where
$$f_n = \sum_{i=9}^{n-9} e_i^+ e_{n-i}^+ \frac{(i-1)\,!\,(n-i-1)\,!}{(n-10)\,!}, \quad \text{for}\ n \geq 18.$$

Using theorem 3.3, (5), we find

$$\big\| F \big\|_n \leq K_4 e_{n+1}^+ (n+1)\,!\,\pi^{-n},$$
$$\big\| F^2 \big\|_n \leq K_3 f_{n+4} (n-6)\,!\,\pi^{-n}. \tag{5.19}$$

Using theorem 4.2, we obtain

$$\big\| \mathcal{C}_2(F) \big\|_n \leq K_5 e_{n+1}^+ (n+1)\,!\,\pi^{-n}. \tag{5.20}$$

Observe that the convergent polynomials series $W_1/2(1-u^2)d^2$, $W_2/2(1-u^2)d^2$ begins with d^9 and $\big(V\,\mathcal{C}_2(Q)+W\,\mathcal{S}_2(Q)\big)/2(1-u^2)d^2$ begin with 1. Using theorem 4.5 we thus obtain

$$\Big\| \frac{W_2}{2(1-u^2)d^2}\mathcal{C}_2(F) \Big\|_n \leq K_6 e_{n-7}^+ (n-7)\,!\,\pi^{-n}, \tag{5.21}$$

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$$\left\|\frac{W_1}{2(1-u^2)d^2}F\right\|_n \le K_7 e_{n-7}^+ (n-7)\,!\pi^{-n}, \tag{5.22}$$

$$\left\|\frac{(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q))}{2(1-u^2)d^2}F^2\right\|_n \le K_8 f_{n+4}^+ (n-6)\,!\pi^{-n}. \tag{5.23}$$

On the other hand

$$\left\|Q_1 J\right\|_n = \left\|\mathcal{C}(G)\right\|_n \le e_n (n-6)\,!\pi^{-n}.$$

We apply the theorems 3.3, (5) and 4.2 and find

$$\left\|\mathcal{C}(J)\right\|_n \le K_9 e_{n+2} (n-4)\,!\pi^{-n}.$$

Because the convergent polynomial series $(W_3 + PV_1)/(1-u^2)d^2 = \mathcal{O}(d^3)$, then

$$\left\|\frac{(W_3 + PV_1)}{(1-u^2)d^2}\mathcal{C}(J)\right\|_n \le K_{10} e_{n-1}^+ (n-9)\,!\pi^{-n}. \tag{5.24}$$

With a crude estimate of the convergent terms $\mu(\varepsilon, u)$ the inequalities (5.21)- (5.24) gives

$$\left\|\frac{V_1}{2(1-u^2)d^2}\mathcal{L}(J)\right\|_n \le K_{11}(1 + e_{n-1}^+ + f_{n+4}^+)(n-6)\,!\pi^{-n}.$$

The multiplication by the convergent terme $2(1-u^2)d^2/V_1$ only changes the constant K_{11}. Since theorem 4.5 applies to the operator $\mathcal{L}$ defined in (4.3), we obtain

$$\left\|dDJ\right\|_n \le K_{12}(1 + e_{n-1}^+ + f_{n+4}^+)(n-6)\,!\pi^{-n}. \tag{5.25}$$

This with theorem 4.5 gives via

$$dD\mathcal{C}(G) = dD(Q_1 J) = J \cdot (dDQ_1) + Q_1 \cdot (dDJ),$$

the estimate

$$\left\|dD\mathcal{C}(G)\right\|_n \le K_{13}(1 + e_{n-3}^+ + f_{n+2}^+)(n-8)\,!\pi^{-n}. \tag{5.26}$$

By equation (5.15), we obtain

$$e_{n-1} \le \frac{K}{n}(1 + e_{n-3}^+ + f_{n+2}^+). \tag{5.27}$$

Lemma 5.2. *Under the condition (5.27), we have $e_n = \mathcal{O}(n^{-1})$ as $n \to \infty$.*

Proof. Let $K_1 \ge 9\,!e_{10}^+$ an arbitrary number. We assume that

$$e_n \le \frac{K_1(n+p)\,!}{(n-1)\,!(p+10)\,!}, \qquad \text{for } 10 \le n \le N-3, \tag{5.28}$$

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with $p \geq -1$, $N \geq 14$. This gives for $13 \leq n \leq N$

$$(n-8)\,!f_{n+2} \leq 2e_9^+ K_1 \frac{8\,!(n+p-7)\,!}{!(p+10)\,!} + K_1^2 \sum_{i=10}^{n-8} \frac{(i+p)\,!(n+p-i+2)\,!}{\left((p+10)\,!\right)^2}.$$

Using the inequality,

$$\sum_{i=10}^{n-8} (i+p)\,!(n+p-i+2)\,! \leq (p+10)\,!(n+p-7)\,!,$$

we obtain

$$f_{n+2} \leq K_2 \frac{(n+p-7)\,!}{(p+10)\,!(n-8)\,!} \leq K_2 \frac{(n+p-3)\,!}{(p+10)\,!(n-4)\,!} \qquad \text{for } 13 \leq n \leq N,$$

with a constant K_2 depends on k_1, independent of p. The asumption (5.27) of the lemma yields

$$e_{n-1} \leq \frac{K}{n}(1 + K_1 + K_2)\frac{(n+p-3)\,!}{(p+10)\,!(n-4)\,!} \qquad \text{for } 13 \leq n \leq N,$$

this implies

$$e_n \leq \frac{K_3}{n+1}\frac{(n+p)\,!}{(p+10)\,!(n-1)\,!}, \qquad \text{for } 12 \leq n \leq N-1,$$

with a constant K_3 only depending upon K_1. Now we choose $N_0 \geq 12$ so large that $\frac{K_3}{N_0} \leq K_1$ and then p so large that (5.28) holds. Then

$$e_n \leq \frac{K_1}{n+1}\frac{(n+p)\,!}{(p+10)\,!(n-1)\,!} \qquad \text{for } 12 \leq n \leq N-1, \text{ with } N \geq N_0.$$

Since $K_1 \geq 9\,!e_{10}$ is an arbitrary number, we have shown for any $p \geq -1$, taht

$$e_n = \mathcal{O}\left(\frac{(n+p)\,!}{(n-1)\,!}\right) \text{ as } n \to \infty,$$

implies that

$$e_n = \mathcal{O}\left(\frac{(n+p-1)\,!}{(n-1)\,!}\right) \text{ as } n \to \infty.$$

Consequently

$$e_n = \mathcal{O}\left(n^{-1}\right), \text{ as } n \to \infty$$

Thus we have shown that

$$\left\|dD\mathcal{C}(G)\right\|_{n+1} = \mathcal{O}\left((n-7)\,!\pi^{-n}\right) \text{ as } n \to \infty .$$

Let $E := \mathcal{C}(G)$. Like G, the polynomial series E is odd in d and the coefficients are odd in u. We partition them

$$
\begin{aligned}
E_n &= \alpha_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u) + \beta_{n-2}(n-3)!\left(\frac{i}{\pi}\right)^{n-3}\tau_{n-2}(u) \\
&\quad + \gamma_{n-4}(n-5)!\left(\frac{i}{\pi}\right)^{n-5}\tau_{n-4}(u) + \overline{E}_{n-6},
\end{aligned}
\tag{5.29}
$$

for odd $n \geq 7$, where α_n and β_n are real number and also $\overline{E}$ have at most degree n for all n. For the whole series E this is equivalent to

$$
\mathcal{C}(G) = E = E_1 + d^2 E_2 + d^4 E_4 + d^6 \overline{E},
\tag{5.30}
$$

where

$$
\begin{aligned}
E_1 &= \sum_{n=7}^{+\infty} \alpha_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)d^n, \\
E_2 &= \sum_{n=5}^{+\infty} \beta_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)d^n, \\
E_3 &= \sum_{n=3}^{+\infty} \gamma_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)d^n, \\
\overline{E} &= \sum_{n=1}^{+\infty} \overline{E}_n(u)d^n.
\end{aligned}
$$

Theorem 5.1 and the definition of the norms yields

$\alpha_n = \mathcal{O}(n^{-7})$, $\beta_n = \mathcal{O}(n^{-5})$, $\gamma_n = \mathcal{O}(n^{-3})$ and $\|D\overline{E}_n\|_{n+1} = \mathcal{O}\big((n-1)!\pi^{-n}\big)$.

We apply $\mathcal{C}^{-1}$ to (5.29) and obtain

$$
G = \mathcal{C}^{-1}(E_1) + d^2\mathcal{C}^{-1}(E_2) + d^4\mathcal{C}^{-1}(E_4) + d^6\mathcal{C}^{-1}(\overline{E}).
\tag{5.31}
$$

To the first two summands, theorem 4.4 applies and yields

$$
\begin{aligned}
\left\|\{\mathcal{C}^{-1}(E_1)\}_n - \alpha(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-7)!\pi^{-n}\big), \\
\left\|\{\mathcal{C}^{-1}(E_2)\}_n - \beta(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-5)!\pi^{-n}\big), \\
\left\|\{\mathcal{C}^{-1}(E_3)\}_n - \gamma(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-3)!\pi^{-n}\big),
\end{aligned}
$$

where

$$
\alpha = \frac{1}{\pi}\sum_{n=7}^{\infty}\alpha_n, \quad \beta = \frac{1}{\pi}\sum_{n=5}^{\infty}\beta_n, \quad \gamma = \frac{1}{\pi}\sum_{n=3}^{\infty}\gamma_n.
$$

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To the last part of (5.29), we apply theorem 4.3 and obtain

$$\left\|\{\mathcal{C}^{-1}(\overline{E})\}_n\right\|_n = \mathcal{O}\big((n-1)!\,\pi^{-n}\log(n)\big).$$

Using $G = \mathcal{C}^{-1}(E)$, we have shown

$$\left\|G_n - \alpha(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\tau_n - \beta(n-3)!\left(\frac{i}{\pi}\right)^{n-3}\tau_{n-2} - \gamma(n-5)!\left(\frac{i}{\pi}\right)^{n-5}\tau_{n-4}\right\|_n$$
$$= \mathcal{O}\Big((n-7)!\,\pi^{-n}\log(n)\Big).$$

If we use (5.3) and the relation $\tau_{n+1} = \frac{1}{n}D\tau_n$ and we apply theorem 3.3, (5), we obtain our final result of this section

$$\left\|F_n - \alpha n!\left(\frac{i}{\pi}\right)^{n+1}\tau'_{n+1} - \beta(n-2)!\left(\frac{i}{\pi}\right)^{n-1}\tau'_{n-1} - \gamma(n-4)!\left(\frac{i}{\pi}\right)^{n-3}\tau'_{n-3}\right\|_n =$$
$$\mathcal{O}\Big((n-5)!\,\pi^{-n}\log(n+2)\Big).$$

and

$$\left\|F_n - \alpha(n+1)!\left(\frac{i}{\pi}\right)^{n+1}\frac{\tau_{n+2}}{\tau_2} - \beta(n-1)!\left(\frac{i}{\pi}\right)^{n-1}\frac{\tau_n}{\tau_2} - \gamma(n-3)!\left(\frac{i}{\pi}\right)^{n-3}\frac{\tau_{n-2}}{\tau_2}\right\|_n$$
$$= \mathcal{O}\Big((n-5)!\,\pi^{-n}\log(n+2)\Big).$$

6 Preparation of the functions to construct a quasi-solutions

In the previous section , we have shown that equation (2.4) has a formal solution and we found an asymptotic approximation of the coefficients of this formal solution. We will use this to construct the quasi-solution. To that purpose, we define the functions

$$H_n(u) := (n+1)!\left(\frac{i}{\pi}\right)^{n+1}\tau_{n+2}(u) \tag{6.1}$$

$$h(t,u) := \sum_{\substack{n=7 \\ n\ odd}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)!}. \tag{6.2}$$

We rewrite

$$h(t,u) = \left(\frac{i}{\pi}\right)^2 \sum_{\substack{n=6 \\ n\ even}}^{\infty} (n+1)(n+2)\left(\frac{it}{\pi}\right)^n \tau_{n+3}(u). \tag{6.3}$$

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Using proposition (3.1)-(4), we obtain

$$h(t, u) = \frac{-1}{\pi^2} \sum_{\substack{n=6 \\ n \ even}}^{\infty} \frac{1}{n!} \left(\frac{it}{\pi}\right)^n \frac{d^n}{d^n\xi} \left(\frac{d^2}{d^2\xi}\left(\tanh(\xi)\right)\right),$$

or equivalently

$$h(t, u) = \frac{-1}{\pi^2} \sum_{\substack{n=6 \\ n \ even}}^{\infty} \frac{1}{n!} \left(\frac{it}{\pi}\right)^n \frac{d^n}{d^n\xi} \left(g(\xi)\right),$$

where $g(\xi) = -2\left(\tanh(\xi) - \tanh(\xi)^3\right)$ and $\xi = \xi(u) = artanh(u)$.
and hence that

$$h(t, u) = \frac{-1}{\pi^2} \sum_{\substack{n=0 \\ n \ even}}^{\infty} \frac{1}{n!} \left(\frac{it}{\pi}\right)^n \frac{d^n}{d^n\xi} \left(g(\xi)\right) + \mu(t, u),$$

where

$$\mu(t, u) := -\frac{2\,u}{\pi^2}\tau_2(u) + \frac{4}{\pi^4}(3u^3 - 2u)\tau_2(u)\,t^2 - \frac{2}{3\pi^6}(45u^5 - 60u^3 + 17u)\tau_2(u)\,t^4.$$

Thus we can write

$$h(t, u) = \frac{-1}{2\pi^2}\left[g\left(\xi + \frac{it}{\pi}\right) + g\left(\xi - \frac{it}{\pi}\right)\right] + \mu(t, u). \tag{6.4}$$

This gives

$$
\begin{aligned}
h(t, u) &= (1 - u^2)\left[\frac{(4u^3 - 4u)\sin\left(\frac{t}{\pi}\right)^4 - (6u^3 - 2u)\sin\left(\frac{t}{\pi}\right)^2 + 2u}{\pi^2\left(\cos\left(\frac{t}{\pi}\right)^2 + u^2\sin\left(\frac{t}{\pi}\right)^2\right)^3}\right] \\
&\quad + \mu(t, u).
\end{aligned}
\tag{6.5}
$$

For fixed real u the function $h(., u)$ is analytic in $|t| < \rho$, where $\rho = \frac{\pi^2}{2}$. In the subsequent definition, we consider real values of u, $0 < u \leq 1$, here $h(., u)$ is also analytic with respect to t on the positive real axis.

Now we define the function $\mathcal{H}(d, u)$ by

$$\mathcal{H}(d, u) := \int_0^{+\infty} e^{-\frac{t}{d}} h(t, u) dt, \quad \text{for} \quad (0 < u \leq 1). \tag{6.6}$$

The function $\mathcal{H}(d, .)$ is real analytic; they can be continued analytically to the interval $-1 < u \leq 1$ in the following way. Choose some positive number M and let Γ_1 the path consisting of the segment from 0 to Mi and of the ray $t \mapsto t + Mi, t \geq 0$.

Let Γ_2 the symmetric path that could also be obtained using $-M$ instead of M. Recalling (6.4), we can also define

$$\mathcal{H}(d,u): \quad = \quad -\frac{1}{\pi^2}\left[\int_{\Gamma_2} e^{-\frac{t}{d}}g(\xi + \frac{it}{\pi})\,dt + \int_{\Gamma_1} e^{-\frac{t}{d}}g(\xi - \frac{it}{\pi})\,dt\right]$$
$$+ \quad \mu_1(d,u),$$

where

$$\mu_1(d,u) := \int_0^\infty e^{-\frac{t}{d}}\mu(t,u)dt,$$

for $-\tanh(\frac{2}{\pi}M) < u \leq 1$, where $\xi = \operatorname{artanh}(u)$, because the singularities of $\tanh$ are $i(\frac{\pi}{2} + n\pi)$, n integer. As M is arbitrary, this defines the analytic continuation of $\mathcal{H}(d,.)$ for $-1 < u \leq 1$.

In the sequel we consider $u_0 \in]-1,0]$.

Lemma 6.1. *If we consider the above function $\mathcal{H}(d,u)$ and the operators defined in (2.22). Then, for $u_0 < u \leq 1$*

    – 1.
$$\mathcal{S}_2(\mathcal{H}) = (1 - u^2)\mathcal{D}_1(d,u),$$

where the function $\mathcal{D}_1(d,u)$ is analytic, beginnings with d^8.

    – 2.
$$\mathcal{C}_2(\mathcal{H}) = -\mathcal{H}(d,u) + (1 - u^2)\mathcal{D}_2(d,u),$$

where the function $\mathcal{D}_2(d,u)$ is analytic, beginnings with d^7.

    – 3
$$\mathcal{S}_2\left(\frac{1}{\tau_2(u)}\mathcal{H}\right) = -\frac{2u\varepsilon}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{D}_3(d,u)$$

where the function $\mathcal{D}_3(d,u)$ is analytic beginnings with d^8

Proof. (1)- If we replace u by T^+ and T^- in (6.5) we obtain

$$\mathcal{H}(d,T^+) - \mathcal{H}(d,T^-) = \int_0^{+\infty} e^{-\frac{t}{d}}\left(h(t,T^+) - h(t,T^-)\right)dt,$$

this implies

$$\mathcal{H}(d,T^+) - \mathcal{H}(d,T^-) = -\frac{1}{2\pi^2}\left(\mathcal{I}^+ - \mathcal{I}^-\right) + \sigma(d,T^+) - \sigma(d,T^-), \qquad (6.7)$$

where

$$\mathcal{I}^+ = \int_0^{+\infty} e^{-\frac{t}{d}}\left(g(\xi(T^+) + \frac{it}{\pi}) + g(\xi(T^+) - \frac{it}{\pi})\right)dt,$$
$$\mathcal{I}^- = \int_0^{+\infty} e^{-\frac{t}{d}}\left(g(\xi(T^-) + \frac{it}{\pi}) + g(\xi(T^-) - \frac{it}{\pi})\right)dt,$$
$$\sigma(d,u) = \int_0^{+\infty} e^{-\frac{t}{d}}\mu(t,u)dt.$$

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Using $\xi(T^{\pm}) = \xi \pm d$, where ξ and u are coupled by $\xi = \mathrm{artanh}(u)$, we obtain

$$\mathcal{I}^+ = \int_0^{+\infty} e^{-\frac{t}{d}} g\!\left(\xi + d + \frac{it}{\pi}\right) dt + \int_0^{+\infty} e^{-\frac{t}{d}} g\!\left(\xi + d - \frac{it}{\pi}\right) dt.$$

If we substitute $t + \pi i \, d$ in the first part, $t - \pi i \, d$ in the second part, we obtain

$$\mathcal{I}^+ = -\int_{-\pi i d}^{+\infty + \pi i d} e^{-\frac{t}{d}} g\!\left(\xi + \frac{it}{\pi}\right) dt - \int_{\pi i d}^{+\infty + \pi i d} e^{-\frac{t}{d}} g\!\left(\xi - \frac{it}{\pi}\right) dt.$$

We apply Cauchy's theorem

$$\mathcal{I}^+ = -\int_0^{+\infty} e^{-\frac{t}{d}} \left(g\!\left(\xi + \frac{it}{\pi}\right) + g\!\left(\xi - \frac{it}{\pi}\right)\right) dt \;+\; \int_0^{-\pi i d} e^{-\frac{t}{d}} g\!\left(\xi + \frac{it}{\pi}\right) dt$$
$$+ \int_0^{\pi i d} e^{-\frac{t}{d}} g\!\left(\xi - \frac{it}{\pi}\right) dt.$$

Substitute $t = -i \, s \, d$ in the second part, $t = i \, s \, d$ in the third part, we obtain

$$\mathcal{I}^+ = -\int_0^{+\infty} e^{-\frac{t}{d}} \left(g\!\left(\xi + \frac{it}{\pi}\right) + g\!\left(\xi - \frac{it}{\pi}\right)\right) dt + 2d \int_0^{\pi} \sin(s) g\!\left(\xi + \frac{s \, d}{\pi}\right) d s.$$

This implies

$$\mathcal{I}^+ \;=\; -\int_0^{+\infty} e^{-\frac{t}{d}} \left(g\!\left(\xi + \frac{it}{\pi}\right) + g\!\left(\xi - \frac{it}{\pi}\right)\right) dt - 4d(1 - u^2)\mathcal{D}^+(d, u), \quad (6.8)$$

where

$$\mathcal{D}^+(d, u) = \int_0^{\pi} \sin(s) \frac{\left(u + \tanh(\frac{s d}{\pi})\right)\left(1 - \tanh(\frac{s d}{\pi})^2\right)}{\left(1 + u \tanh(\frac{ds}{\pi})\right)^3} d s.$$

We can also use the same method for $\mathcal{I}^-$ and obtain

$$\mathcal{I}^- \;=\; -\int_0^{+\infty} e^{-\frac{t}{d}} \left(g\!\left(\xi + \frac{it}{\pi}\right) + g\!\left(\xi - \frac{it}{\pi}\right)\right) dt - 4d(1 - u^2)\mathcal{D}^-(d, u), \quad (6.9)$$

where

$$\mathcal{D}^-(d, u) = \int_0^{\pi} \sin(s) \frac{\left(u - \tanh(\frac{ds}{\pi})\right)\left(1 - \tanh(\frac{s d}{\pi})^2\right)}{\left(1 - u \tanh(\frac{s d}{\pi})\right)^3} d s.$$

Consequently

$$\mathcal{H}(d, T^+) - \mathcal{H}(d, T^-) \;=\; \frac{2d(1 - u^2)}{\pi^2}\left(\mathcal{D}^+(d, u) - \mathcal{D}^-(d, u)\right) + \sigma(d, T^+) - \sigma(d, T^-).$$

Using (2.22), we find

$$\mathcal{S}_2(\mathcal{H}) = (1 - u^2)\,\mathcal{D}_1(d, u), \qquad\qquad (6.10)$$

where $\mathcal{D}_1(d,u)$ is analytic, beginnings with d^8.

(2). Using the same method, we obtain

$$\mathcal{H}(d,T^+) + \mathcal{H}(d,T^-) = -2\mathcal{H}(d,u) + 2(1-u^2)\mathcal{D}_2(d,u).$$

With (2.22) we find

$$\mathcal{C}_2(\mathcal{H}) = -\mathcal{H}(d,u) + (1-u^2)\mathcal{D}_2(d,u).$$

$\mathcal{D}_2(d,u)$ is analytic, beginnings with d^7.

(3). Using (5.7), (2.8), (2.22) and (1), (2) of this lemma, we obtain

$$\begin{aligned}
\mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\mathcal{H}\Big) &= \mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\Big)\mathcal{C}_2(\mathcal{H}) + \mathcal{C}_2\Big(\frac{1}{\tau_2(u)}\Big)\mathcal{S}_2(\mathcal{H}) \\
&= -\frac{2u\varepsilon}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{D}_3(d,u)
\end{aligned}$$

where

$$\mathcal{D}_3(d,u) := \big(1 + (1+u^2)\sinh(d)^2\big)(1-u^2)\mathcal{D}_1(d,u) + 2u\varepsilon(1-u^2)\mathcal{D}_2(d,u).$$

Proposition 6.2. *We have*

1. *For $u_0 < u \le 1$*

$$\mathcal{H}(d,u) \sim \sum_{\substack{n=7 \\ n\ odd}}^{\infty} (n+1)!\Big(\frac{i}{\pi}\Big)^{n+1}\tau_{n+2}(u)d^n, \quad as\ d \searrow 0, \tag{6.11}$$

2. $\big|\frac{\partial\mathcal{H}}{\partial u}(d,u)\big| \le Kd \quad for \ u_0 \le u \le 1 \ \ (d>0).$

Proof.

1. To prove (1) we use Watson's lemma and (6.2). if $|t - k\pi^2| < \frac{\pi^2}{2}$, for $u_0 \le u < 1$

2. – For $u_0 \le u \le \frac{1}{2}$

$$\big|\mathcal{H}(d,u)\big| = \frac{1}{d^6}\int_0^\infty e^{-t/d}h^{(-6)}(t,u)\,dt,$$

where $h^{(-6)}(t,u)$ satisfies

$$\Big(\frac{\partial}{\partial t}\Big)^6 h^{(-6)} = \frac{\partial h}{\partial u} \quad and \Big(\frac{\partial}{\partial t}\Big)^k h^{(-6)}(0,u) = 0, \quad for\ k = 0, ..., 5.$$

Because of

$$h^{(-6)}(t,u) \le K\frac{t^7}{7!} \qquad for\ t \ge 0,$$

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we obtain

$$\left|\frac{\partial \mathcal{H}}{\partial u}(d, u)\right| \leq \frac{K}{7!d^6} \int_0^\infty e^{-\frac{t}{d}} t^7 dt.$$

This implies

$$\left|\frac{\partial \mathcal{H}}{\partial u}(d, u)\right| \leq Kd \qquad (d > 0).$$

– for $\frac{1}{2} \leq u \leq 1$

$$\begin{aligned}
\left|\frac{\partial \mathcal{H}}{\partial u}(d, u)\right| &= \left|\int_0^\infty e^{-\frac{t}{d}} \frac{\partial h}{\partial u}(d, u) dt\right|, \\
&\leq K \int_0^\infty e^{-\frac{t}{d}} dt, \\
&\leq Kd \qquad (d > 0). \ \square
\end{aligned}$$

Let us consider a sequence $R_n(u)$ of polynomials of degree at most n, such that

$$\|R_n\|_n = O\Big((n-5)!\pi^{-1}\log(n)\Big)$$

Lemma 6.3. *Let polynomial series difine $R \in \mathcal{Q}$ with the above estimate be given and define*

$$\begin{aligned}
r(t, u) : &= \sum_{n=7}^\infty R_n(u) \frac{t^{n-1}}{(n-1)!}, \qquad (t \in \mathbb{C}, |t| \leq \frac{\pi^2}{2}, u_0 \leq u \leq 1), \\
r(t, u) : &= r\left(\frac{\pi^2}{2}, u\right) + \left(t - \frac{\pi^2}{2}\right) \frac{\partial r}{\partial t}\left(\frac{\pi^2}{2}, u\right), \qquad (t > \frac{\pi^2}{2}, u_0 \leq u \leq 1), \\
\mathcal{R}(d, u) : &= \int_0^\infty e^{-\frac{t}{d}} r(t, u) dt.
\end{aligned}$$

Then

1. *r is continuously differentiable function on the set B of all (t, u) such that u satisfies $u_0 \leq u \leq 1$ and t is a complex number and satisfies $|t| \leq \frac{\pi^2}{2}$ or $t > \frac{\pi^2}{2}$. The restriction of r to $u_0 \leq u \leq 1, |t| \leq \frac{\pi^2}{2}$ is twice continuously differentiable. For fixed $u_0 \leq u \leq 1$ the function $r(t, u)$ is analytic in $|t| < \frac{\pi^2}{2}$.*

2. *$\mathcal{R}(d, u)$ is continuous, partially differentiable with respect to u, has continuous partial derivative and*

$$\mathcal{R}(d, u) \sim \sum_{n=7}^\infty R_n(u) d^n, \ as \ d \searrow 0. \tag{6.12}$$

3. *$\left|\mathcal{R}(d, u)\right| \leq Kd^3, \ \left|\frac{\partial \mathcal{R}}{\partial u}(d, u)\right| \leq Kd^3, \quad for \ \ u_0 \leq u \leq 1 \ \ (d > 0).$*

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The importance of our definition of $\mathcal{R}$ lies in a certain compatibility with insertion of the functions T^+, T^- for u. First let

$$\sum_{n=7}^{\infty} R_n^+(u)d^n \;\; := \;\; \sum_{n=7}^{\infty} R_n\big(T^+(d,u)\big)d^n,$$

$$\sum_{n=7}^{\infty} R_n^-(u)d^n \;\; := \;\; \sum_{n=7}^{\infty} R_n\big(T^-(d,u)\big)d^n.$$

We obtain a new sequences $R_n^+(u), R_n^-(u)$ of polynomials of degree at most n. Theorem 4.2, and (4.2) imply

$$\big\|R_n^+(u)\big\|_n \;\; = \;\; O\Big((n-5)\,!\pi^{-n}\log(n)\Big)$$

$$\big\|R_n^-(u)\big\|_n \;\; = \;\; O\Big((n-5)\,!\pi^{-n}\log(n)\Big)$$

Therefore we can use lemma 6.3 for $R_n^+(u)$, $R_n^-(u)$, and obtain functions $\mathcal{R}^+(d,u)$, $\mathcal{R}^-(d,u)$.

Theorem 6.4. *There is a positive constant K independent of d, u such that*

$$\big|\mathcal{R}^+(d,u) - \mathcal{R}(d,T^+)\big| \;\; \leq \;\; Kd^3 e^{-\frac{\pi^2}{2d}} \quad for \quad (0 < d < d_0, u_0 < u < 1)$$

$$\big|\mathcal{R}^-(d,u) - \mathcal{R}(d,T^-)\big| \;\; \leq \;\; Kd^3 e^{-\frac{\pi^2}{2d}} \quad for \quad (0 < d < d_0, u_0 < u < 1).$$

Proof The proof is exactly the one of [5].

Definition 6.5. *Let $\mathcal{D}(d,u)$ be a function defined for $0 < d < d_0$ and $u_0 < u < 1$. We say that $\mathcal{D}(d,u)$ has property G if*

$$\mathcal{D}(d,u) = \int_0^\infty e^{-\frac{t}{d}} q(t,u)dt \quad (0 < d < d_0, \; u_0 < u < 1)$$

is the Laplace transform of some function $q(t,u)$ that has the following proprties :

1. *$q(t,u)$ is defined if $u_0 < u < 1$ and either t is complex and $|t| < \frac{\pi^2}{2}$ or t is real and $t \geq 0$,*

2. *$q(t,u)$ is analytic in $|t| < \frac{\pi^2}{2}$ for $u_0 < u < 1$,*

3. *$q(t,u)$ restricted to $0 \leq t < \frac{\pi^2}{2}$ or $t \geq \frac{\pi^2}{2}$ is continuous and the $\lim_{t \to \frac{\pi^2}{2}} q(t,u)$ exists for every $u_0 < u < 1$,*

4. *there is a positive constant K such that*

$$|q(t,u)| \leq Ke^{Kt}, \quad for \;\; t \geq 0, \; u_0 < u < 1, \; (0 < d < d_0, \; u_0 < u < 1)$$

The set of all function with the property G will be denote by $\mathcal{G}$

Lemma 6.6. *For $u_0 < u \le 1$*

1. *If $\mathcal{H}(d, u)$ is the function defined in (6.6) then*

$$d^4 \mathcal{H}(d, u) = (1 - u^2)\tilde{\mathcal{H}}(d, u) + \mathcal{O}\left((1 - u^2)e^{-\frac{\pi^2}{2d}}\right)$$

   where $\tilde{\mathcal{H}}(d, u)$ has property G

2. *Let k be a psitive integer. If $\mathcal{D}_1, \mathcal{D}_2$ have property G and their first terms in the Taylor development at $d = 0$, begin with d^k then*

$$\mathcal{D}_1(d, u)\mathcal{D}_2(d, u) = d^k \mathcal{D}(d, u) + \mathcal{O}\left(d^k e^{-\frac{\pi^2}{2d}}\right),$$

   where $\mathcal{D}(d, u)$ has property G

3. *If $\mathcal{H}(d, u)$ is the function defined in (6.6) then*

$$\mathcal{H}(d, u)^2 = (1 - u^2)^2 d^3 \mathcal{E}(d, u) + \mathcal{O}\left((1 - u^2)^2 d^3 e^{-\frac{\pi^2}{2d}}\right)$$

   where $\mathcal{E}(d, u)$ has property G

4. *Any function $\mathcal{D}(d, u)$ analytic in a neighborhood of $d = 0$ has property G if $\mathcal{D}(0, u) = 0$ for all u,*

5. *If $\mathcal{R}(d, u)$ is defined by lemma 6.3 then $\dfrac{1}{d^2}\mathcal{R}(d, u)$ has property G*

6. *If $\mathcal{D}_1, \mathcal{D}_2$ have property G then so do $\mathcal{D}_1 + \mathcal{D}_2$, $\mathcal{D}_1 - \mathcal{D}_2$ and $\mathcal{D}_1 \cdot \mathcal{D}_2$*

7. *If $\mathcal{D}(d, u)$ has property G then*

$$\left|\mathcal{D}(d, u)\right| \le Kd \quad \left(0 < d < \frac{1}{K}\right) \tag{6.13}$$

   with some constant $K > 0$ independent of u.

Proof.

1. For $u > 0$, we have

$$d^4 \mathcal{H} = (1 - u^2) \int_0^\infty e^{-\frac{t}{d}} g_4(t, u)dt \tag{6.14}$$

   where
$$g_4(t, u) = \frac{1}{(1 - u^2)} \int_0^t \int_0^\theta \int_0^\upsilon \int_0^\tau h(s, u)ds\, d\tau\, d\upsilon\, d\theta$$

   $g_4(t, u)$ has a logarithmic singularity at $t_k(s) = (2k + 1)\dfrac{\pi^2}{2} \pm d\dfrac{\pi s}{\varepsilon}i$ for $(k \ge 0, s > 0)$. it is analytic in $|t| < \frac{\pi^2}{2}$ and $\lim_{t \to \frac{\pi^2}{2}} g_4(t, u)$ exists.

   If we put
$$\tilde{\mathcal{H}}(d, u) = \int_0^\infty e^{-\frac{t}{d}} \tilde{g}_4(t, u)dt$$

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where

$$\tilde{g}_4(t,u) = \begin{cases} g_4(t,u), & \text{if } t \leq \frac{\pi^2}{2} \\ g_4\left(\frac{\pi^2}{2},u\right), & \text{if } t \geq \frac{\pi^2}{2} \end{cases}$$

then $\tilde{\mathcal{H}}(d,u)$ has property G and

$$d^4\mathcal{H}(d,u) = (1-u^2)\tilde{\mathcal{H}}(d,u) + \mathcal{O}\left((1-u^2)e^{-\frac{\pi^2}{2d}}\right).$$

(ii)- For $-u_0 < u < 1$, where $0 < u_0 < 1$, we have

$$\begin{aligned}
\mathcal{H}(d,u) &= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h(t,u) dt + 2\pi i \sum_{k\geq 0} Res\left(e^{-\frac{t}{d}} h(t,u), t_k(s)\right) \\
&= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h(t,u) dt + \mathcal{O}\left(\frac{1}{d^2}(1-u^2)e^{-\frac{\pi^2}{2d}}\right)
\end{aligned}$$

where $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{4}$. For $0 < u < u_0$, this formula coincides with the formula

$$\int_0^{\infty} e^{-\frac{t}{d}} h(t,u) dt$$

and extends it by real analytic continuation for $-u_0 < u < 0$.
This implis

$$d^4\mathcal{H}(d,u) = (1-u^2) \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} g_4(t,u) dt + \mathcal{O}\left((1-u^2)d^2 e^{-\frac{\pi^2}{2d}}\right)$$

we obtain

$$\begin{aligned}
d^4\mathcal{H}(d,u) &- (1-u^2)\tilde{\mathcal{H}}(d,u) = (1-u^2)\int_\Gamma e^{-\frac{t}{d}} g_2(t,u) d\Gamma \\
&+ \mathcal{O}\left((1-u^2)e^{-\frac{\pi^2}{2d}}\right),
\end{aligned}$$

where Γ is the path following the real line from infinity to $(\frac{\pi^2}{2},0)$, then along the vertical line from $(\frac{\pi^2}{2},0)$ to $\left(\frac{\pi^2}{2},\frac{\pi^2}{2}\tan(\varphi)\right)$ and finally along the line $y = \tan(\varphi)x$ from $\left(\frac{\pi^2}{2},\frac{\pi^2}{2}\tan(\varphi)\right)$ to infinity.
Since $g_4(.,u)$ is bounded on Γ, then

$$\left|d^4\mathcal{H}(d,u) - (1-u^2)\tilde{\mathcal{H}}(d,u)\right| \leq K(1-u^2)e^{-\frac{\pi^2}{2d}},$$

where K is positive constant. Finally

$$d^4\mathcal{H}(d,u) = (1-u^2)\tilde{\mathcal{H}}(d,u) + \mathcal{O}\left((1-u^2)e^{-\frac{\pi^2}{2d}}\right) \quad \text{for} \quad (0 < u_0 < u \leq 1)$$

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2. We assume that $\mathcal{D}_1(d, u), \mathcal{D}_2(d, u)$ have property G and their first terms in the Taylor development at $d = 0$ begin with d^k. Then

$$\mathcal{D}_1 = \int_0^\infty e^{-\frac{t}{d}} f(t, u) dt$$

$$\mathcal{D}_2 = \int_0^\infty e^{-\frac{t}{d}} g(t, u) dt$$

where $f(t, u), g(t, u)$ are analytic in $|t| < \frac{\pi^2}{2}$ and $f(t, u) = O(t^{k-1}), g(t, u) = O(t^{k-1})$.

$$\mathcal{D}_1(d, u) \mathcal{D}_2(d, u) = \int_0^\infty e^{-\frac{t}{d}} (f * g)(t, u) dt \qquad (6.15)$$

Since

$$
\begin{aligned}
h(t, u) & = (f * g)(t, u) = \int_0^s f(t, u) g(t - s, u) ds \\
& = \int_0^t f(t - s, u) g(s, u) ds \\
& = \int_{t/2}^t f(s, u) g(t - s, u) ds + \int_{t/2}^t f(t - s, u) g(s, u) ds.
\end{aligned}
$$

For $t < \frac{\pi^2}{2}$, the function $h(t, u)$ is k times differentiable with respect to t and

$$
\begin{aligned}
h'(t, u) & = f(t, u) g(0, u) + f(0, u) g(t, u) - f(\frac{t}{2}, u) g(\frac{t}{2}, u) \\
& + \int_{t/2}^t f(s, u) g'(t - s, u) ds + \int_{t/2}^t f'(t - s, u) g(s, u) ds \\
& = \int_{t/2}^t f(s) g'(t - s) ds + \int_{t/2}^t f'(t - s, u) g(s, u) ds \\
& - f(\frac{t}{2}, u) g(\frac{t}{2}, u) \\
h^{(k)}(t, u) & = \int_{t/2}^t f(s) g^{(k-1)}(t - s) ds + \int_{t/2}^t f^{(k-1)}(t - s, u) g(s, u) ds \\
& - \sum_{n=0}^{k-1} f^{(n)}(\frac{t}{2}, u) g^{(k-1-n)}(\frac{t}{2}, u)
\end{aligned}
$$

Observe that $h^{(k)}(t, u)$ is continuous on $[0, \pi^2[$, it is analytic for $|t| < \frac{\pi^2}{2}$. If we put

$$
\tilde{h}(t, u) = \begin{cases} h(t, u), & \text{if } t < \frac{\pi^2}{2} \\ \left(t - \frac{\pi^2}{2}\right)^k. & \text{if } t \geq \frac{\pi^2}{2} \end{cases}
$$

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then
$$\int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u)dt,$$

has property G and

$$
\begin{aligned}
(\mathcal{D}_1 \cdot \mathcal{D}_2)(d,u) &= \int_0^\infty e^{-\frac{t}{d}} h(t,u)dt = \int_0^\infty e^{-\frac{t}{d}} \tilde{h}(t,u)dt + \mathcal{O}\left(d^k e^{-\frac{\pi^2}{2d}}\right) \\
&= d^k \int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u)dt + \mathcal{O}\left(d^k e^{-\frac{\pi^2}{2d}}\right) \\
&= d^k \mathcal{D}(d,u) + \mathcal{O}\left(d^k e^{-\frac{\pi^2}{2d}}\right)
\end{aligned}
$$

where $\mathcal{D}(d,u)$ has property G.

3. We have
$$\mathcal{H}(d,u)^2 = \frac{1}{d^8}(d^4\mathcal{H})\cdot(d^4\mathcal{H})$$

Using (1) and (2) of this lemma, we obtain immediately the result.

For $0 \le u \le 1$, the proof of (4), (5), (6) and (7) is exactly the one of [5]. This proof is valid for $u_0 < u \le 1$.

7 Approximate solution $\mathcal{A}(d,u)$ of the functional equation (2.4)

In this section we use the formal solution of (2.4). The estimates of section 5 for its coefficients and the preliminary results of section 6 allow to construct the function $\mathcal{A}(d,u)$ that almost satisfies (2.4), except an error which is exponentially small as $d \to 0$.

In theorem 2.1 we found that (2.4) has a uniquely determined formal power series solution

$$A(d,u) = \frac{1}{\varepsilon}B(d,u) = \frac{1}{\varepsilon}\left(U + \sum_{\substack{n=7 \\ n \text{ odd}}}^\infty F_n(u)d^n\right), \tag{7.1}$$

where

$$
\begin{aligned}
U(d,u) &= \varepsilon u + (u - u^3)d^3 + (\frac{10}{3}u^5 - \frac{16}{3}u^3 + 2u)d^5, \\
F_n &= \alpha n!\left(\frac{i}{\pi}\right)^{n+1}\tau'_{n+1}(u) + \beta(n-2)!\left(\frac{i}{\pi}\right)^{n-1}\tau'_{n-1}(u) \\
&\quad + \gamma(n-4)!\left(\frac{i}{\pi}\right)^{n-4}\tau'_{n-3}(u) + R_n(u), \\
\|R_n\|_n &= O\left((n-5)!\pi^{-1}\log(n)\right), \quad (n \ge 7).
\end{aligned}
$$

Consequently

$$B(d,u) = \left(\alpha + \beta d^2 + \gamma d^4\right) \sum_{\substack{n=7 \\ n\,\mathrm{odd}}}^{\infty} (n+1)!\left(\frac{i}{\pi}\right)^{n+1}\frac{\tau_{n+2}(u)}{\tau_2(u)}d^n + \sum_{\substack{n=7 \\ n\,\mathrm{odd}}}^{\infty} R_n(u)d^n + U(d,u),$$

Now we set

$$\mathcal{B}(d,u) = \left(\alpha + \beta d^2 + \gamma d^4\right)\frac{1}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{R}(d,u) + U(d,u), \quad (d > 0, u_0 < u < 1),$$

(7.2)

where $\mathcal{H}(d,u)$ is defined in (6.6) and $\tilde{\mathcal{R}}(d,u)$ is the function corresponding to R_n, $n = 7, 9, ..$ By lemma 6.3 and proposition 6.2, we have

$$\mathcal{B}(d,u) \sim B(d,u), \quad \text{as } d \searrow 0 \text{ for every} \quad u_0 < u < 1. \tag{7.3}$$

Theorem 7.1. *We have*

$$\left|\mathcal{A}(d,T^+) - \mathcal{A}(d,T^-) - f\left(\varepsilon, \mathcal{A}(d,u)\right)\right| \leq \frac{K}{d}e^{-\frac{\pi^2}{2d}}, \quad \text{for } (0 < d < d_0, u_0 \leq u < 1),$$

where K is a constant independent of d, u and

$$\mathcal{A}(d,u) = \frac{1}{\varepsilon}\mathcal{B}(d,u) \tag{7.4}$$

Proof. We set

$$\mathcal{T}(d,u) = \mathcal{S}_2(\mathcal{B}) - \varepsilon^2 + \mathcal{B}(d,u)^2. \tag{7.5}$$

then

$$\begin{aligned}
\mathcal{T}(d,u) &= \left(\alpha + \beta d^2 + \gamma d^4\right)\mathcal{S}_2\left(\frac{1}{\tau_2(u)}\mathcal{H}\right) - \varepsilon^2 + \mathcal{B}(d,u)^2 \\
&+ \frac{1}{2}\mathcal{R}(d,T^+) - \frac{1}{2}\mathcal{R}(d,T^-) + \mathcal{S}_2(U)
\end{aligned}$$

With (7.2) and lemma 6.1, this implies

$$\mathcal{T}(d,u) = \mathcal{T}_0(d,u) \cdot \mathcal{H}(d,u) + \mathcal{T}_1(d,u) + \mathcal{T}_2(d,u),$$

where

$$\begin{aligned}
\mathcal{T}_0(d,u) &= \left(\alpha + \beta d^2 + \gamma d^4\right)\left[-2\varepsilon u + 2U(d,u)\right]\frac{1}{\tau_2(u)}, \\
\mathcal{T}_1(d,u) &= \left(\alpha + \beta d^2 + \gamma d^4\right)^2\frac{1}{\tau_2(u)^2}\mathcal{H}(d,u)^2 + \mathcal{R}(d,u)^2 \\
&+ 2\left(\alpha + \beta d^2 + \gamma d^4\right)\frac{1}{\tau_2(u)}\mathcal{H}(d,u)\mathcal{R}(d,u) \\
\mathcal{T}_2(d,u) &= 2U(d,u)\mathcal{R}(d,u) + \frac{1}{2}\mathcal{R}(d,T^+) - \frac{1}{2}\mathcal{R}(d,T^-), \\
\mathcal{T}_3(d,u) &= U(d,u)^2 + \mathcal{S}_2(U) - \varepsilon^2 + \frac{d}{2}\mathcal{D}(d,u)
\end{aligned}$$

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Observing that $\mathcal{T}_0(d, u)$ is analytic function beginnings with d^3. Using (1) of lemma 6.6, we obtain that $d\mathcal{T}_0(d, u)\mathcal{H}(d, u)$ has property G.

Applying lemma 6.6, we obtain

$$\mathcal{T}_1(d, u) = d^3 \mathcal{D}_3(d, u) + \mathcal{O}\left(d^3 e^{-\frac{\pi^2}{2d}}\right)$$

where $\mathcal{D}_3(d, u)$ has property G.

Using (5) of lemma 6.6 and theorem 6.4, we find

$$\mathcal{T}_2(d, u) = d^3 \mathcal{D}_4(d, u) + \mathcal{O}\left(d^3 e^{-\frac{\pi^2}{2d}}\right)$$

where $\mathcal{D}_4(d, u)$ has property G.

On the other hand, $\frac{1}{d}\mathcal{T}_3(d, u)$ has property G. This implies

$$\mathcal{T}(d, u) = \frac{1}{d}\mathcal{D}_5(d, u) + \mathcal{K}(d, u),$$

where $\mathcal{D}_5(d, u)$ has property G and

$$\left|\mathcal{K}(d, u)\right| \le K d^3 e^{\frac{-\pi^2}{2d}} \quad \text{for} \quad (0 < d < d_0, \ u_0 < u < 1). \tag{7.6}$$

Altogether we have proved that

$$\mathcal{T}(d, u) = \frac{1}{d}\int_0^\infty e^{-\frac{t}{d}}\delta(t, u)dt + \mathcal{K}(d, u),$$

where $\delta(., u)$ is analytic in $|t| < \frac{\pi^2}{2}$, it is continuous on $[0, \frac{\pi^2}{2}[$ and $]\frac{\pi^2}{2}, \infty[$ and has a limit as $t \to \frac{\pi^2}{2}$ for every $u_0 < u < 1$ and satisfaies

$$|\delta(t, u)| \le K\, e^{Kt}, \tag{7.7}$$

with a constant K independent of u. Now if $\delta(t, u) = \sum_{n=0}^\infty \delta_n(t)t^n$ is the power series of $\delta(t, u)$ near $t = 0$, Watson's lemma implies with (7.6) that

$$d\mathcal{T}(d, u) \sim \sum_{n=0}^\infty n!\delta_n d^{n+1}, \quad \text{as } d \searrow 0 \text{ for every } u_0 < u < 1.$$

On the other hand, because of its definition

$$\begin{aligned}
\mathcal{T}(d, u) \quad &\sim \quad \frac{1}{2}B(d, T^+) - \frac{1}{2}B(d, T^-) - \varepsilon^2 + B(d, u)^2 \\
&\sim \quad 0 + 0d + \cdots,
\end{aligned}$$

because the formal series B satisfy (5.1). But this means that all δ_n must vanish and $\delta(t, u) \equiv 0$ if
$|t| < \frac{\pi^2}{2}$, $u_0 < u < 1$. With (7.7), thus we obtain for $u_0 < u < 1$

$$\int_0^\infty e^{-\frac{t}{d}}|\delta(t, u)|dt \le \int_{\frac{\pi^2}{2}}^\infty e^{-\frac{t}{d}}Ke^{Kt}dt \le Kde^{-\frac{\pi^2}{2d}}, \quad (0 < d < d_0).$$

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We obtain
$$\left|\mathcal{T}(d,u)\right| \le Ke^{-\frac{\pi^2}{2d}} \quad (0 < d < d_0).$$

Finally, we have shown that there is a constant K such that

$$\left|\mathcal{A}(d,T^+) - \mathcal{A}(d,T^-) - f\left(\varepsilon, \mathcal{A}(d,u)\right)\right| = \frac{1}{\varepsilon}\left|\mathcal{T}(d,u)\right| \le \frac{K}{d}e^{-\frac{\pi^2}{2d}},$$

for $0 < d < d_0$, $0 \le u < 1$.

8 Distance between points of manifolds

Let $y(t), t \in\,]-1, \infty[$ an exact solution to the difference equation (2.1), so that the stable manifold W_s^+ is parametrized by $t \mapsto \left(y(t), y(t+\varepsilon)\right)$. We know that $y^-(t) = -y^-(-t)$ is also an exact solution of (2.1) with $y^-(t) \to 1$ as $t \to\ -\infty$. Morever the instable manifold W_i^- is parametrized by $t \mapsto \left(y^-(t), y^-(t+\varepsilon)\right)$. We introduced the quasi-solution $\mathcal{A}(d,u)$ in previous section, then $\mathcal{A}^-(d,u) = -\mathcal{A}(d,-u)$ is a quasi-solution close to the exact solution $y^-(t)$. We denote by $\tilde{W}_s$, $\tilde{W}_i$ the manifolds close to W_s^+ respectively W_i^- parametrized by $t \mapsto \left(\xi_+(t), \xi_+(t+\varepsilon)\right)$ respectively $t \mapsto \left(\xi_-(t), \xi_-(t+\varepsilon)\right)$, where $\xi_+(t) = \mathcal{A}\left(d, u(t)\right)$ and $\xi_-(t) = \mathcal{A}^-\left(d, u(t)\right)$.

In this section we will show that the vertical distance between a point of the stable manifold and the manifold $\tilde{W}_s$ is exponentially small as well as that between $\tilde{W}_i$ and W_i^- ; both are smaller, as we will show than the distance between $\tilde{W}_s$ and $\tilde{W}_i$. This eventually proves that W_s^+ and W_i^- do not coincide.

In order to estimate the distance between some point (x_0, y_0) on the stable manifold W_s^+, we consider the sequence $P_n = (x_n, y_n)$, defined by $P_{n+1} = \phi(P_n)$, where

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + 2\varepsilon(1-y^2) \end{pmatrix}. \tag{8.1}$$

$P_n = (x_n, y_n)$ is a point of the stable manifold. We obtain

$$P_{n+1} = \phi\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ x_n + 2\varepsilon(1-y_n^2) \end{pmatrix} \tag{8.2}$$

Since $\Phi = \phi \circ \phi$, where Φ is defined in (1.3), and the stable manifold W_s^+ is invariant by the application Φ, then the stable manifold W_s^+ remains invariant by ϕ. Then theorem 7.1 implies

$$\xi_+(t+\varepsilon) = \xi_+(t-\varepsilon) + 2\varepsilon\left(1-\xi_+(t)^2\right) + \frac{1}{\varepsilon}\eta(t)e^{-\frac{\pi^2}{2\varepsilon}} \tag{8.3}$$

where $\eta(t)$ is bounded. There is a sequence $t_n \in\,]-1, +\infty[$ such that

$$\xi_+(t_n - \varepsilon) = x_n = \mathcal{A}\left(d, T^-\left(d, u(t_n)\right)\right) = \mathcal{A}\left(d, T^-\left(d, U_\varepsilon(n)\right)\right) \tag{8.4}$$
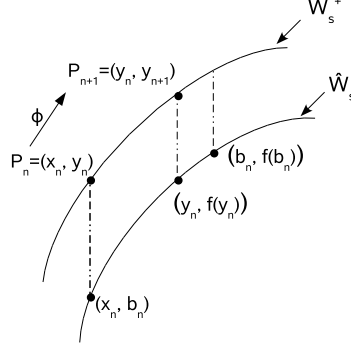
FIG. 4 – The vertical distance between a point of the stable manifold and the approximation defined using the quasi-solution $\mathcal{A}$ for the logistic equation .

$$
\begin{aligned}
\xi_+(t_n) &= \mathcal{A}\big(d, U_\varepsilon(n)\big) \\
\xi_+(t_n + \varepsilon) &= \mathcal{A}\Big(d, T^+\big(d, U_\varepsilon(n)\big)\Big)
\end{aligned}
\tag{8.5}
$$

where $U_\varepsilon(n) = u(t_n)$. The relation (8.4) implies that the point (x_n, b_n), where $b_n = \xi_+(t_n)$, the vertical projection of the point (x_n, y_n) on the curve $t \mapsto \big(\xi_+(t), \xi_+(t + \varepsilon)\big)$. But, according to (8.2), (8.4), we have

$$
\xi_+(t_{n+1} - \varepsilon) = x_{n+1} = y_n.
\tag{8.6}
$$

Now consider the error quantities

$$
d_\varepsilon(n) = y_n - \mathcal{A}\big(d, U_\varepsilon(n)\big) = y_n - b_n,
\tag{8.7}
$$

$d_\varepsilon(n)$ denotes the vertical distance between a point P_n and the manifold $\hat{W}_s$. The manifold $\tilde{W}_s^+$ is parameterized by the curve
$t \mapsto \big(\xi_+(t), \xi_+(t + \varepsilon)\big)$. We obtain

$$
d_\varepsilon(n + 1) = y_{n+1} - f(x_{n+1}) = y_{n+1} - f(y_n) = y_{n+1} - f\big(d_\varepsilon(n) + b_n\big),
\tag{8.8}
$$

where $f(x) = \xi_+\Big(\xi_+^{-1}(x) + \varepsilon\Big)$. The point $\big(y_n, f\big(y_n\big)\big)$ is the vertical projection of the point (x_{n+1}, y_{n+1}) on the curve $t \mapsto \big(\xi_+(t), \xi_+(t + \varepsilon)\big)$ (See FIG.4).
If we use Taylor expainson, (8.8) implies

$$
d_\varepsilon(n + 1) = y_{n+1} - f(b_n) - f'(\theta_n)d_\varepsilon(n), \qquad b_n < \theta_n < b_n + d_\varepsilon(n).
\tag{8.9}
$$

Then according to the definitions of f and b_n, we have $f(b_n) = \xi_+(t_n + \varepsilon)$, with (8.3) and (8.4), we obtain

$$
f(b_n) = x_n + 2\varepsilon(1 - b_n^2) + \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}.
\tag{8.10}
$$

Together with (8.9) and (8.2) this implies

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$$d_\varepsilon(n+1) = -f'(\theta_n)d_\varepsilon(n) + 2\varepsilon(b_n^2 - y_n^2) - \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}, \tag{8.11}$$

Using (8.7) we obtain

$$d_\varepsilon(n+1) = -\Big(f'(\theta_n) + 2\varepsilon(y_n + b_n)\Big)d_\varepsilon(n) - \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}. \tag{8.12}$$

Now

$$f'(\theta_n) = \frac{\xi'_+(z_n + \varepsilon)}{\xi'_+(z_n)}, \quad \text{where} \quad z_n = \xi_+^{-1}(\theta_n),$$

then

$$f'(\theta_n) = 1 - 2u(z_n)d + \big(3u(z_n)^2 - 1\big)d^2 + O(d^3).$$

On the other hand $b_n < \theta_n < y_n$ implies

$$0 < b_n < \xi(z_n) < y_n$$

and thus

$$\begin{aligned} f'(\theta_n) + 2\varepsilon(y_n + b_n) \quad &> \quad 1 + 2\varepsilon\Big(\xi_+(z_n) + b_n\Big) - 2u(z_n)d \\ &\quad + \Big(3u(z_n)^2 - 1\Big)d^2 + O(d^3). \end{aligned} \tag{8.13}$$

We know that

$$\xi_+(z_n) = u(z_n) + \Big(u(z_n) - u(z_n)^3\Big)d^2 + O(d^4),$$

this with (2.8) implies

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) > 1 + 2\varepsilon b_n + O(d^2).$$

As $b_n > 0$, then for sufficiently small d,

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) > 1.$$

This implies

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) = 1 + g_\varepsilon\big(u(z_n)\big), \quad \text{with some} \quad g_\varepsilon\big(u(z_n)\big) \geq \varepsilon c,$$

$g_\varepsilon\big(u(z_n)\big) = O(\varepsilon)$ and c is a positive constant. With (8.12), this implies

$$d_\varepsilon(n+1) = -\Big(1 + g_\varepsilon\big(u(z_n)\big)\Big)d_\varepsilon(n) - \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}. \tag{8.14}$$

As $d_\varepsilon(n) \to 0$ as $n \to +\infty$, this implies that

$$|d_\varepsilon(n)| \leq \frac{1}{\varepsilon}e^{-\frac{\pi^2}{2\varepsilon}}\sum_{k=n}^{\infty}\left|\frac{\eta(s_k)}{\prod_i^{i=k}\Big(1 + g_\varepsilon\big(u(z_k)\big)\Big)}\right|, \tag{8.15}$$

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provided this series converges. This follows from

$$\sum_{k=n}^{\infty} \left| \frac{\eta(s_k)}{\prod_i^{i=k}\left(1 + g_\varepsilon\big(u(z_k)\big)\right)} \right| \leq K_1 \sum_{k=0}^{+\infty}(1 + \varepsilon c)^{-k},$$

where we used that η is bounded. This imply that there is a constant $K > 0$ such that

$$\sum_{k=n}^{\infty} \left| \frac{\eta(s_k)}{\prod_i^{i=k}\left(1 + g_\varepsilon\big(u(z_k)\big)\right)} \right| \leq \frac{K}{\varepsilon}.$$

With (8.5) implies

$$d_\varepsilon(n) = O\left(\frac{1}{\varepsilon^2}\exp\left(-\frac{\pi^2}{2\varepsilon}\right)\right),$$

and in particular that the vertical distance $d_\varepsilon(0)$ of (x_0, y_0) to $\tilde{W}_s$ is

$$d_\varepsilon(0) = O\left(\frac{1}{\varepsilon^2}\exp\left(-\frac{\pi^2}{2\varepsilon}\right)\right). \tag{8.16}$$

Now we will calculate the vertical distance between the twomanifolds $\tilde{W}_s$ and $\tilde{W}_i$. To do this, we will need to extend both quasi-solutions in an adequate way. We define

$$\begin{aligned}
\tilde{\xi}_+(t) &= \mathcal{A}\big(d, u(t)\big), \quad \text{for} \quad -\frac{1}{2} \leq u \leq 1, \\
\tilde{\xi}_-(t) &= \mathcal{A}^-\big(d, u(t)\big), \quad \text{for} \quad -1 \leq u \leq \frac{1}{2}.
\end{aligned} \tag{8.17}$$

First, we evaluate the quantities

$$d_\varepsilon\left(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\right) = \mathcal{A}(d, u) - \mathcal{A}^-(d, u), \text{ for } -\frac{1}{2} \leq u \leq \frac{1}{2} \tag{8.18}$$

using (7.2) and (7.4)

$$\begin{aligned}
d_\varepsilon\left(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\right) &= \tilde{\xi}_+(t) - \tilde{\xi}_-(t), \tag{8.19}\\
&= \frac{1}{\varepsilon}(\alpha + \beta d^2 + \gamma d^4)\big(\tilde{\mathcal{H}}(d, u) - \tilde{\mathcal{H}}^-(d, u)\big),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{H}}(d, u) &= \int_0^\infty e^{-\frac{s}{d}}h(s, u)\,ds, \quad \text{for} \quad -\frac{1}{2} \leq u \leq \frac{1}{2}, \\
\tilde{\mathcal{H}}^-(d, u) &= -\int_0^\infty e^{-\frac{s}{d}}h(s, -u)\,ds, \quad \text{for} \quad -\frac{1}{2} \leq u \leq \frac{1}{2}. \tag{8.20}
\end{aligned}$$

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Modifying the path of integral, we obtain

$$\tilde{\mathcal{H}}(d, u) = \int_{\Gamma} e^{-\frac{s}{d}} h(s, u) \, ds \; + \sum_{Im(s_k(u))<0} 2\pi i Res\Big(e^{-\frac{s}{d}} h(s, u), s_k(t)\Big)$$

$$- \sum_{Im(s_k(u))>0} 2\pi i Res\Big(e^{-\frac{s}{d}} h(s, u), s_k(t)\Big),$$

where $s_k(t) = \frac{\pi^2}{2} \pm \frac{d\pi t}{\varepsilon} i + k\pi^2$ for $k \geq 0$, Γ is the sum of two paths Γ_1 and Γ_2. Here Γ_1 consist of two segments of the higher half-plane, one of these two segments is parallel to the axis $y = 0$ and begins at the point $(\frac{\pi^2}{4}, 1)$, the other ends in the point $(0,0)$, and the path Γ_2 is the symmetry of Γ_1 relatively to the axis $y = 0$.
 This implies

$$\tilde{\mathcal{H}}(d, u) = \int_{\Gamma} e^{-\frac{s}{d}} h(s, u) \, ds \; + \; \frac{\pi e^{\frac{\pi t}{\varepsilon} i}}{2d^2(1 - u^2)} \sum_{k=0}^{\infty} \exp\Big(-\frac{(k+1)\pi^2}{2d}\Big)$$

$$+ \; \frac{\pi e^{-\frac{\pi t}{\varepsilon} i}}{2d^2(1 - u^2)} \sum_{k=0}^{\infty} \exp\Big(-\frac{(k+1)\pi^2}{2d}\Big)$$

Therefore

$$\tilde{\mathcal{H}}(d, u) = \int_{\Gamma} e^{-\frac{s}{d}} h(s, u) \, ds + \frac{\pi \cos(\frac{\pi t}{\varepsilon})}{d^2(1 - u^2)} \sum_{k=0}^{\infty} \exp\Big(-\frac{(k+1)\pi^2}{2d}\Big). \qquad (8.21)$$

Similary for $\tilde{\mathcal{H}}^-(d, u)$

$$\tilde{\mathcal{H}}^-(d, u) = \int_{\Gamma} e^{-\frac{s}{d}} h(s, u) \, ds - \frac{\pi \cos(\frac{\pi t}{\varepsilon})}{d^2(1 - u^2)} \sum_{k=0}^{\infty} \exp\Big(-\frac{(k+1)\pi^2}{2d}\Big).$$

Therefore

$$d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big) = \frac{2\pi(\alpha + \beta d^2 + \gamma d^4)\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3 \, (1 - u^2)} \exp\Big(-\frac{\pi^2}{2d}\Big) + \mathcal{O}\Big(\frac{\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3} e^{-\frac{3\pi^2}{2d}}\Big),$$

for $-\text{arctanh}(\frac{1}{2}) < t < \text{arctanh}(\frac{1}{2})$. Consequently

$$d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big) = \frac{2\pi\alpha \cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3 \big(1 - \tanh(\frac{d}{\varepsilon}t)^2\big)} e^{-\frac{\pi^2}{2\varepsilon}} + \mathcal{O}\Big(\frac{1}{\varepsilon}\cos(\frac{\pi t}{\varepsilon})e^{-\frac{\pi^2}{2d}}\Big) \qquad (8.22)$$

 Now we want to estimate the vertical distance between a point on the quasi-manifold $\tilde{W}_s$ and its vertical projection on the quasimanifold $\tilde{W}_i$. For this purpose let t such that $-\frac{1}{2} \leq u(t) \leq \frac{1}{2}$. Then the point $\Big(\tilde{\xi}_+(t), \tilde{\xi}_+(t + \varepsilon)\Big)$ is on the quasimanifold $\tilde{W}_s$. We suppose that $\Big(\tilde{\xi}_+(t), \tilde{\xi}^-(t_1 + \varepsilon)\Big)$, where $-\frac{1}{2} \leq u(t_1) \leq \frac{1}{2}$, is

its vertical projection on the quasimanifold $\tilde{W}_i$, then the vertical distance between these two points is

$$dist_v(t) = \tilde{\xi}_+(t+\varepsilon) - \tilde{\xi}_-(t_1+\varepsilon) = \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_+(t)\big), \tag{8.23}$$

where $\tilde{f}(x) = \tilde{\xi}_-\big(\tilde{\xi}_-^{-1}(x) + \varepsilon\big)$, $\tilde{\xi}_-^{-1}$ the inverse function of $\tilde{\xi}_-$. Using (8.22) and (8.20) we obtain

$$dist_v(t) = \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_-(t) + e_\varepsilon(t)\big),$$

where $e_\varepsilon(t) = d_\varepsilon\big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\big)$. We use Taylor expansion and obtain

$$\begin{aligned} dist_v(t) &= \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_-(t)\big) - e_\varepsilon(t) \cdot \tilde{f}'(\theta), \\ &= \tilde{\xi}_+(t+\varepsilon) - \tilde{\xi}_-(t+\varepsilon) - e_\varepsilon(t) \cdot \tilde{f}'(\theta), \end{aligned} \tag{8.24}$$

where $\tilde{\xi}_-(t) < \theta < \tilde{\xi}_-(t) + e_\varepsilon(t)$. With (8.20) this implies

$$dist_v(t) = e_\varepsilon(t+\varepsilon) - e_\varepsilon(t) \cdot \tilde{f}'(\theta). \tag{8.25}$$

Since $e_\varepsilon(t+\varepsilon) = -e_\varepsilon(t) + \mathcal{O}\big(\varepsilon \cdot e_\varepsilon(t)\big)$ and $\tilde{f}'(\theta) = \dfrac{\tilde{\xi}_-'\big(\tilde{\xi}_-^{-1}(\theta) + \varepsilon\big)}{\tilde{\xi}_-'\big(\tilde{\xi}_-^{-1}(\theta)\big)} = 1 + \mathcal{O}(\varepsilon)$,

we obtain

$$dist_v(t) = \big(-2 + \mathcal{O}(\varepsilon)\big) e_\varepsilon(t). \tag{8.26}$$

Consequently

$$dist_v(t) \sim -\frac{4\pi\alpha \cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3\big(1 - \tanh(\frac{d}{\varepsilon}t)^2\big)} e^{-\frac{\pi^2}{2\varepsilon}}. \tag{8.27}$$

Combining (8.16) and (8.27), we conclude

$$Dist_\varepsilon\big(w_s^+(t), W_i^-\big) = \frac{4\pi\alpha \cos(\frac{\pi}{\varepsilon}t + \pi)}{\varepsilon^3\big(1 - \tanh(\frac{d}{\varepsilon}t)\big)} e^{\frac{-\pi^2}{2\varepsilon}} + O\left(\frac{1}{\varepsilon^2} e^{\frac{-\pi^2}{2\varepsilon}}\right), \quad \text{as } \varepsilon \searrow 0,$$

9   An estimate for the main asymptotic coefficients

In this section we show that the number α in the main theorem of the introduction is not zero and we want to find bounds for α_n and will then obtain estimats for α using the first α_n which can be computed explicitly. We recall that

$$\alpha = \frac{1}{\pi} \sum_{n=7}^{\infty} \alpha_n \tag{9.1}$$

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where $\alpha_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}u^n$ is the leading term of the coefficient $\{\mathcal{C}(G)\}_n(u)$ of d^n in $\mathcal{C}(G)(d,u)$, where $G = Q(d,u).F(d,u)$, $F(d,u) = B(d,u) - U(d,u)$ (cf.(5.2) and (5.5)).

For a polynomial series $X(d,u) = \sum_{n=0}^{\infty} X_n(u)d^n$ where the degrees of $X_n(u)$ do not exceed n, we can write $X_n(u) = \sum_{k=0}^{n} x_{nk}u^n$. Then we denote by $\widehat{X}(z) = \sum_{n=0}^{\infty} x_{nn}z^n$.

The mapping $X(d,u) \mapsto \widehat{X}(z)$ extracts the leading terms of the series $X(d,u)$. It is compatible with addition and multiplication. If we define the following operator $\widehat{D} = -z^2\frac{d}{dz}$, we also find

$$
\begin{aligned}
\widehat{D}\widehat{X}(z) &= \widehat{DX}(z) = -z^2\widehat{X}'(z) \\
\widehat{\mathcal{C}(X)}(z) &= \cosh\left(\frac{\widehat{D}}{2}\right)\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\left(\frac{2z}{2+z}\right) + \widehat{X}\left(\frac{2z}{2-z}\right)\right) \\
\widehat{\mathcal{S}(X)}(z) &= \sinh\left(\frac{\widehat{D}}{2}\right)\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\left(\frac{2z}{2+z}\right) - \widehat{X}\left(\frac{2z}{2-z}\right)\right) \\
\widehat{\mathcal{C}_2(X)}(z) &= \cosh(\widehat{D})\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\left(\frac{z}{1+z}\right) + \widehat{X}\left(\frac{z}{1-z}\right)\right) \\
\widehat{\mathcal{S}_2(X)}(z) &= \sinh(\widehat{D})\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\left(\frac{z}{1+z}\right) - \widehat{X}\left(\frac{z}{1-z}\right)\right)
\end{aligned}
$$

where the operators $\mathcal{S}, \mathcal{C}, \mathcal{S}_2, \mathcal{C}_2$ are defined in (2.18). If we apply the mapping $X(d,u) \mapsto \widehat{X}(z)$ to the equation (5.13) , we obtain

$$
\widehat{\mathcal{L}}\left(\frac{\widehat{E}}{\widehat{Q}_1}\right)(z) = \widehat{\mathcal{R}}(\widehat{E})(z), \tag{9.2}
$$

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where

$$\widehat{\mathcal{L}}(x) \;:=\; -\frac{\widehat{V}_1}{z^2}\left(\widehat{\mathcal{S}}(x) - \frac{\widehat{\mathcal{S}}(z^2)}{\widehat{\mathcal{C}}(z^2)}\widehat{\mathcal{C}}(x)\right)$$

$$\widehat{\mathcal{R}}(x) \;:=\; -\frac{\widehat{\mu}}{z^2} + \frac{\widehat{W}_5}{z^2}\widehat{\mathcal{C}}\left(\frac{x}{\widehat{Q}_1}\right) - \frac{\widehat{W}_1}{2z^2\widehat{Q}}\mathcal{Y} + \frac{\widehat{W}_4}{2z^2\widehat{Q}^2}\widehat{\mathcal{C}}\left(\frac{x}{\widehat{Q}_1}\right)\mathcal{Y}^2$$

$$-\;\frac{\widehat{W}_2}{2z^2}\widehat{\mathcal{C}}_2\left(\frac{\mathcal{Y}}{\widehat{Q}}\right),$$

$$\widehat{E} \;=\; \widehat{\mathcal{C}(G)} = \sum_{n=0\,\mathrm{odd}}^{\infty} \alpha_n(n-1)!\left(\frac{i}{\pi}\right)^{n-1}z^n,$$

$$\mathcal{Y} \;=\; \widehat{\mathcal{C}}^{-1}(x),$$

$$\widehat{Q} \;=\; -z^2 + z^4 - \frac{13}{6}z^6 + \frac{47}{18}z^8,$$

$$\widehat{Q}_1 \;=\; -z^2 - \frac{3}{2}z^4,$$

$$\widehat{V}_1 \;=\; 4z^2\Big[-192 - 240z^2 + 508z^4 + 1995z^6 + 3418z^8$$

$$+\; 4389z^{10} + 217z^{12}\Big]\Big/\Big[3(z^2-4)^4\Big],$$

$$\widehat{W}_2 \;=\; z^{11}\Big[591 - 15457z^2 + 21610z^4 + 59194z^6 + 36583z^8$$

$$+\; 2759z^{10}\Big]\Big/\Big[27(z^2-1)^8\Big],$$

$$\widehat{W}_1 \;=\; z^{11}\Big[2385 - 32832z^2 - 11790z^4 + 231917z^6 - 116267z^8 + 720682z^{10}$$

$$+\; 1018936z^{12} + 290089z^{14} + 2480z^{16}\Big]\Big/\Big[81(z^2-1)^8\Big],$$

$$\widehat{W}_4 \;=\; \widehat{V}\widehat{\mathcal{C}}_2(\widehat{Q}) + \widehat{W}\widehat{\mathcal{S}}_2(\widehat{Q})$$

$$=\; -z^2\Big[54 - 432z^2 + 1449z^4 - 2724z^6 + 4449z^8 - 14719z^{10} - 12857z^{12}$$

$$+\; 95146z^{14} + 110905z^{16} + 29041z^{18} + 248z^{20}\Big]\Big/\Big[54(z^2-1)^8\Big],$$

$$\widehat{W}_5 \;=\; \widehat{W}_3 + \widehat{P}\widehat{V}_3,$$

$$=\; -8z^5\Big[-1152 + 1168z^2 + 9744z^4 + 18901z^6 + 14664z^8$$

$$+\; 1240z^{10}\Big]\Big/\Big[3(z^2-4)^4(4+z^2)\Big],$$

$$\widehat{\mu} = z^{10}\Big[-527000z^{30} - 1988300z^{28} + 14763570z^{26} - 25488145z^{24}$$

$$+ \quad 11808105z^{22} + 11381027z^{20} - 17065562z^{18} - 15479497z^{16} - 15777878z^{14}$$

$$- \quad 21499321z^{12} + 14844222z^{10} + 4847178z^8 - 1419003z^6 - 541026z^4$$

$$+ \quad 320490z^2 - 50220\Big]\Big/\Big[972(z^2-1)^{12}\Big], \tag{9.3}$$

with $x = x(z)$ is a power series.

Now for integers $m > p \geq -1$, $m \geq 0$, we introduce the set

$$\mathcal{M}(m,p) = \Big\{ \sum_{n=m}^{\infty}{}^* x_n z^n \backslash\ x_n\big/\big[(n-p)\,!\,\pi^{-n}\big], n = m, m+2... \text{ is bounded}\Big\}$$

$\mathcal{M}(m,p)$ is a Banach space with the norm

$$|x|_{m,p} = \sup\big\{ |x_n|\big/\big[(n-p)\,!\,\pi^{-n}\big], n \geqq m, n \equiv m \quad \mathrm{mod}\ 2\big\}. \tag{9.4}$$

Here the symbol $\displaystyle\sum_{n=m}^{\infty}{}^*$ for the summation means that only summands with $n \equiv m \ \ \mathrm{mod}\ 2$ are considered.

Proposition 9.1. *Let m,p,q be integers and $m \geq p \geq -1, m \geq q \geq -1, m \geq 0$. The norms (9.4) have the following property :*

1. *If $q < p \leq m$ then $\mathcal{M}(m,p) \subset \mathcal{M}(m,q)$ and*

$$|x|_{m,q} \leq \frac{(m-p)\,!}{(m-q)\,!}|x|_{m,p} \quad for\ x \in \mathcal{M}(m,p) \tag{9.5}$$

2. *If $x \in \mathcal{M}(m,p)$ then $zx \in \mathcal{M}(m+1,p+1)$ and*

$$|zx|_{m+1,p+1} = \pi|x|_{m,p} \tag{9.6}$$

3. *$\widehat{D} : \mathcal{M}(m,p) \mapsto \mathcal{M}(m+1,p)$ and*

$$\pi|x|_{m,p} \leq |\widehat{D}(x)|_{m+1,p} \leq \pi\frac{m}{m-p+1}|x|_{m,p} \tag{9.7}$$

Now we consider an even power series $f(z) = \sum_{j=0}^{\infty}{}^*f_j z^j$ having a radius of convergence greater than π. Then $f(\widehat{D})$ maps $\mathcal{M}(m,p)$ into itself and

$$|f(\widehat{D})x|_{m,p} \leq \|f\|_{m,p}|x|_{m,p}, \quad \text{for}\ x \in \mathcal{M}(m,p) \tag{9.8}$$

where

$$\|f\|_{m,p} = \sup\Big\{ \sum_{j=0}^{n-m}{}^*|f_j|\pi^j \prod_{k=1}^{p-1}\frac{n-k}{n-j-k}\backslash n \geq m, n \equiv m \quad \mathrm{mod}\ 2\Big\} \text{ if } p \geq 1,$$

$$\|f\|_{m,p} = \sup\Big\{ \sum_{j=0}^{\infty}{}^*|f_j|\pi^j\backslash n \geq m, n \equiv m \quad \mathrm{mod}\ 2\Big\} \text{ if } p \leq 0. \tag{9.9}$$

We have also

$$|f.x|_{m,p} \le M_{m,p}(f)|x|_{m,p}, \quad \text{for } x \in \mathcal{M}(m,p), \qquad (9.10)$$

where

$$M_{m,p}(f) = \sup\left\{ \sum_{j=0}^{n-m}\!^* |f_j|\pi^j \frac{(n-p-j)\,!}{(n-p)\,!} \backslash n \ge m, n \equiv m \mod 2 \right\}. \qquad (9.11)$$

We conclude that the inverse of $\widehat{\mathcal{C}}$ maps $\mathcal{M}(m,p)$ with $p \ge 3$ into $\mathcal{M}(m,1)$ and

$$|\widehat{\mathcal{C}}^{-1}(x)|_{m,1} \le c_{m,p}|x|_{m,p}, \text{ with } c_{m,p} = \frac{4}{\pi} \sum_{n=m}^{\infty}\!^* \frac{(n-p)\,!}{(n-1)\,!}. \qquad (9.12)$$

Let $x \in \mathcal{M}(m,p)$ and $y \in \mathcal{M}(m,q)$ and assume that $m + q \le n + p$. Then we obtain $x.y \in \mathcal{M}(n+m,q+m)$ and

$$|x.y|_{n+m,q+m} \le \alpha_{m-p,n-q}|x|_{m,p}|y|_{n,q}, \qquad (9.13)$$

with

$$\alpha_{r,s} = \sup\left\{ \sum_{j=r}^{N-s}\!^* \frac{j\,!(N-j)\,!}{(N-r)\,!} \backslash N \ge r+s, N-r-s \text{ even} \right\}.$$

Finally we need estimates for the inverse of the linear operator $\widehat{\mathcal{L}}$ in (9.2). $\widehat{\mathcal{L}}$ maps $\mathcal{M}(m-1,q)$ onto $\mathcal{M}(m,q)$ and is one to one. Now let $x \in \mathcal{M}(m-1,p)$ and $w = \widehat{\mathcal{L}}(x)$. If we proceed similar to the proof of theorem 4.6, we find that

$$\begin{aligned} \widehat{D}y &= f(\widehat{D})\left[-\frac{(4+z^2)}{4\widehat{V}_1}w\right], \\ x &= z^2 y. \end{aligned} \qquad (9.14)$$

where $f(z) = z/\sinh(z/2)$.

We can apply (9.6)..(9.13) and obtain for $m \ge \max(p+1,4)$

$$|\widehat{\mathcal{L}}^{-1}(w)|_{m-1,p} \le L_{m,p}|w|_{m,p} \quad \text{for } w \in \mathcal{M}(m,p), \qquad (9.15)$$

where

$$L_{m,p} = \frac{1}{\pi}\left\|\frac{z}{\sinh(z/2)}\right\|_{m-2,p-2} M_{m,p}\left(\frac{(4+z^2)z^2}{4\widehat{V}_1}\right). \qquad (9.16)$$

Now we are in a position to use (9.2) to estimate $\widehat{E}$. We use that $\widehat{E}$ can be computed recursively from (9.2). If x is an odd power series and $x \equiv \widehat{E} \mod z^n$ with $n \ge 9$ then via $\widehat{\mathcal{C}}^{-1}(x) \equiv \widehat{G} \mod z^n$ we first obtain $\widehat{\mathcal{R}}(x) \equiv \widehat{\mathcal{R}}(\widehat{E}) \mod z^{n+1}$

and then $\widehat{\mathcal{L}}^{-1}\left(\widehat{\mathcal{R}}(x)\right) \equiv \frac{\widehat{E}}{\widehat{Q}_1}$ mod z^{n+1}. Thus if $x_0 \equiv 0$ mod z^9 is arbitrary and we let

$$x_{n+1} := \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\widehat{\mathcal{R}}(x_n)\right) \quad \text{for } n = 0... \tag{9.17}$$

then $x_n \equiv \widehat{E}$ mod z^{9+2n} for all n. We begin with the polynomial x_0 of degree ≤ 79 with $x_0 \equiv \widehat{E}$ mod z^{81}. We computed it explicitly and found that

$$|x_0|_{9,7} \leq K_0 := 308027.359777894414. \tag{9.18}$$

Then we consider the sequence x_n of polynomial series defined in (9.17), which now satisfies $x_n \equiv \widehat{E}$ mod z^{81+2n}. We will show

$$|x_n|_{9,7} \leq K_0 \quad \text{for all } n \geq 0 \tag{9.19}$$

and this implies $|\widehat{E}|_{9,7} \leq K_0$ because each coefficient of $\widehat{E}$ is also that of some x_n if n is sufficiently large.

We prove (9.19) by induction. For $n = 0$ it is true. Assume that $|x_n|_{9,7} \leq K_0$, for this Let us take again the equation (9.18) with (9.2). We obtain

$$
\begin{aligned}
x_{n+1} \quad + \quad & \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right) = \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_5}}{z^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\right) - \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_1}}{2z^2\widehat{Q}}\mathcal{Y}_n\right) \\
+ \quad & \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_4}}{2z^2\widehat{Q}^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\mathcal{Y}_n^2\right) - \widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_2}}{2z^2}\widehat{\mathcal{C}}_2\left(\frac{\mathcal{Y}_n}{\widehat{Q}}\right)\right) \tag{9.20}
\end{aligned}
$$

where $\mathcal{Y}_n = \widehat{\mathcal{C}}^{-1}(x_n)$. Then using (9.2)..(9.17), we obtain

$$\left|\widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_5}}{z^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\right)\right|_{11,10} \leq K_1|x_0|_{9,7},$$

$$\left|\widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_1}}{2z^2\widehat{Q}}\mathcal{Y}_n\right)\right|_{17,10} \leq K_2|x_0|_{9,7},$$

$$\left|\widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_2}}{2z^2}\widehat{\mathcal{C}}_2\left(\frac{\mathcal{Y}_n}{\widehat{Q}}\right)\right)\right|_{17,10} \leq K_3|x_0|_{9,7},$$

$$\left|\widehat{Q}_1 \widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_4}}{2z^2\widehat{Q}^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\mathcal{Y}_n^2\right)\right|_{15,10} \leq K_4|x_0|_{9,7}^2$$

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where

$$K_1 : = \pi^2 M_{9,8}\left(1+\frac{3}{2}z^2\right)L_{10,8}M_{7,5}\left(\frac{\widehat{W_5}}{z^5}\right)M_{9,7}\left(\frac{z^2}{\widehat{Q_1}}\right)\left\|\cosh\left(\frac{z}{2}\right)\right\|_{7,5},$$

$$K_2 : = \pi^9 M_{15,8}\left(1+\frac{3}{2}z^2\right)L_{16,8}M_{9,1}\left(\frac{\widehat{W_1}}{2z^9\widehat{Q}}\right)c_{9,7},$$

$$K_3 : = \pi^9 M_{15,8}\left(1+\frac{3}{2}z^2\right)L_{16,8}M_{7,-1}\left(\frac{\widehat{W_2}}{2z^{11}}\right)\left\|\cosh(z)\right\|_{7,-1}M_{9,1}\left(\frac{z^2}{\widehat{Q}}\right)c_{9,7},$$

$$K_4 : = \frac{1}{56\pi^2}M_{13,8}\left(1+\frac{3}{2}z^2\right)L_{14,8}M_{14,8}\left(\frac{\widehat{W_4}}{2z^2}\right)M_{18,12}\left(\frac{z^4}{\widehat{Q}^2}\right)\alpha_{8,8}\ c_{9,7}^2.$$

This imply

$$\left|x_{n+1}+\widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right)\right|_{11,10}\le\left(K_1+K_2+K_3+K_4K_0\right)K_0$$

Evaluation of these quantities yields approximately

K_1	K_2	K_3	K_4
375142.8501573	633.1622891589	10214.44411459	0.000011150281

and altogether

$$\left|x_{n+1}+\widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right)\right|_{11,10}\le 118896679183. \tag{9.21}$$

Now we consider $g(z):=z^2\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right)$, letting $z^2y=z^{-2}g(z)$, with (9.15) we obtain

$$\widehat{D}y=\widehat{D}\left(z^{-4}g\right)=f\left(\widehat{D}\right)\left[\frac{\widehat{\mu}(z)(4+z^2)}{4z^4\widehat{V}_1}\right], \tag{9.22}$$

where

$$f(z):=\frac{z}{\sinh(z/2)}.$$

The coefficient of z^k in $f(z)$ is smaller than $4(2\pi)^k$ we obtain by comparison of the coefficients of z^j in (9.22)

$$(j-1)|g_{j+3}|\le 4\sum_{k=4}^{j}*(2\pi)^{k-j}\,h_k\frac{(j-1)!}{(k-1)!}$$

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and hence

$$\frac{|g_j|\pi^j 2^j}{(j-2)!} \leq 4\sum_{k=4}^{\infty} \frac{h_k(2\pi)^k}{(k-1)!} \leq 3668333.$$

We obtain for add $j \geq 9$

$$\frac{|g_j|}{(j-10)!\pi^{-j}} \leq \pi^3 \cdot 3668333 \cdot (j-9)(j-8)(j-7)(j-6)(j-5)2^{-j+3}.$$

and

$$\frac{|g_{j-2}|}{(j-10)!\pi^{-j}} \leq \pi^5 \cdot 3668333 \cdot (j-9)(j-8)(j-7)2^{-j+5}.$$

but, $\left\{\frac{\widehat{Q}_1}{z^2}g(z)\right\}_j = g_j + \frac{3}{2}g_{j-2}$, with (9.20) this yields eventually

$$\frac{\left|\{x_{n+1}\}_j\right|}{(j-10)!\pi^{-j}} \leq 118896679184 \quad \text{for } j \geq 81.$$

This implies

$$\frac{\left|\{x_{n+1}\}_j\right|}{(j-7)!\pi^{-j}} \leq \frac{118896679184}{72 \cdot 73 \cdot 74} \leq 305691 \quad \text{for } j \geq 81$$

and with $x_{n+1} \equiv x_n \mod z^{81}$ we finally proved that $|x_{n+1}|_{9,7} \leq K_0 = 308027.35$. Thus the proof of (9.18) is complete.

Theorem 9.2.

$$|\alpha_n| \leq \frac{98048.15}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)} \quad \text{for } n \geq 9.$$

and even better

$$|\alpha_n| \leq 37845988419\frac{(n-10)!}{(n-1)!} \quad \text{for odd } n \geq 81.$$

As a consequence we obtain estimates for α

$$
\begin{aligned}
\left|\alpha - \frac{1}{\pi}\sum_{n=1}^{N}{}^*\alpha_n\right| &\leq& 12046752267\sum_{n=N}^{\infty} \frac{(n-10)!}{(n-1)!} \quad \text{for } n \geq 9, \\
&\leq& 12046752267 \cdot \frac{(N-10)!}{16(N-2)!}\left(1+\frac{8}{N-1}\right), \\
&\leq& \frac{762333542}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}
\end{aligned}
$$

We evaluated α_n for $n = 9, 11, ..., 81$ and obtain

$$|\alpha - 1.264150331| \leq 6 \cdot 10^{-7}. \tag{9.23}$$

Références

[1] W. Eckhaus, Asymptotic Analysis of Singular perturbations, North-Holland, Amsterdam (1979).

[2] A. Fruchard, R. Schäfke, Exponentially small splitting of separatrices for difference equations with small step size, Journal of Dynamical and Control Systems, Volume 2, Number 2 / April, 1996, 193-238.

[3] V. Hakim, K. Mallick, Exponentially small splitting of separatrices, matching in the complex plan and Borel summation, Nonlinearity 6(1993) 57-70.

[4] V.F. Lazutkin,I.G. Schachmannski and M.B. Tabanov, Splitting of separatrices for standard and semistandard mappings, Physica D, 40 235-248, (1989).

[5] R. Schäfke, H. Volkmer, Asymptotic analysis of the equichordal problem, J.reine u. Angew. Math. 425(1992), 9-60.